

# SEMI-INVARIANTS FOR CONCEALED-CANONICAL ALGEBRAS

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ABSTRACT. In the paper is we generalize known descriptions of rings of semi-invariants for regular modules over Euclidean and canonical algebras to arbitrary concealed-canonical algebras.

Throughout the paper  $\mathbb{k}$  is a fixed algebraically closed field. By  $\mathbb{Z}$ ,  $\mathbb{N}$  and  $\mathbb{N}_+$  we denote the sets of the integers, the non-negative integers and the positive integers, respectively. Finally, if  $i, j \in \mathbb{Z}$ , then  $[i, j] := \{k \in \mathbb{Z} \mid i \leq k \leq j\}$  (in particular,  $[i, j] = \emptyset$  if  $i > j$ ).

## INTRODUCTION

Concealed-canonical algebras have been introduced by Lenzing and Meltzer [22] as a generalization of Ringel's canonical algebras [26]. An algebra is called concealed-canonical if it is isomorphism to the endomorphism ring of a tilting bundle over a weighted projective line. The concealed-canonical algebras can be characterized as the algebras which posses sincere separating exact subcategory [23] (see also [28]). Together with tilted algebras [7, 20], the concealed-canonical algebras form two most prominent classes of quasi-tilted algebras [19]. Moreover, according to a famous result of Happel [18], every quasi-tilted algebra is derived equivalent either to a tilted algebra or to a concealed-canonical algebra.

Despite investigations of a structure of the categories of modules over concealed-canonical algebras, geometric problems have been studied for this class of algebras (see for example [2, 3, 6, 14, 15, 17, 29]). Often these problem were studied for canonical algebras only and sometimes the authors restrict their attention to the concealed-canonical algebras of tame representation type.

In the paper we study a problem, which has been already investigated in the case of canonical algebras. Namely, given a concealed-canonical algebra  $\Lambda$  and a module  $R$ , which is a direct sum of modules from of sincere separating exact subcategory of  $\text{mod } \Lambda$ , we want to describe a structure of the ring of semi-invariants associated to  $\Lambda$  and the dimension vector of  $R$ . This problem has been solved provided  $\Lambda$  is a canonical algebra and  $R$  comes from a distinguished sincere separating

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exact subcategory of  $\text{mod } \Lambda$  (the answers have been obtained independently by Skowroński and Weyman [29] and Domokos and Lenzing [14, 15]). This problem has also been solved for another class of concealed-canonical algebras, namely the path algebras of Euclidean quivers [30] (see also [12, 27]). The obtained results are very similar, although the methods used in the proof are completely different. The aim my paper is to obtain a unified proof of the above results, which would generalize to an arbitrary concealed-canonical algebra. This aim is achieved if the characteristic of  $\mathbb{k}$  equals 0. If  $\text{char } \mathbb{k} > 0$ , then we show that an analogous result is true if we study the semi-invariants which are the restrictions of the semi-invariants on the ambient affine space. The precise formulation of the obtained results can be found in Section 6. In particular we prove that the studied rings of semi-invariants are always complete intersections, and are polynomial rings if the considered dimension vector is “sufficiently big”.

The paper is organized as follows. In Section 1 we introduce a setup of quivers and their representations, which due to a result of Gabriel [16] is an equivalent way of thinking about algebras and modules. Next, in Section 2 we gather facts about concealed-canonical algebras (equivalently, quivers). In Section 3 we introduce semi-invariants and present their basic properties. Next, in Section 4 we study the semi-invariants in the case of concealed-canonical quivers more closely. Section 5 is devoted to presentation of necessary facts about the Kronecker quiver, which is the minimal concealed-canonical quiver. Finally, in Section 6 we present and proof the main result.

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## 1. QUIVERS AND THEIR REPRESENTATIONS

By a quiver  $\Delta$  we mean a finite set  $\Delta_0$  (called the set of vertices of  $\Delta$ ) together with a finite set  $\Delta_1$  (called the set of arrows of  $\Delta$ ) and two maps  $s, t : \Delta_1 \rightarrow \Delta_0$ , which assign to each arrow  $\alpha$  its starting vertex  $s\alpha$  and its terminating vertex  $t\alpha$ , respectively. By a path of length  $n \in \mathbb{N}_+$  in a quiver  $\Delta$  we mean a sequence  $\sigma = (\alpha_1, \dots, \alpha_n)$  of arrows such that  $s\alpha_i = t\alpha_{i+1}$  for each  $i \in [1, n-1]$ . In the above situation we put  $\ell\sigma := n$ ,  $s\sigma := s\alpha_n$  and  $t\sigma := t\alpha_1$ . We treat every arrow in  $\Delta$  as a path of length 1. Moreover, for each vertex  $x$  we have a trivial path  $\mathbf{1}_x$  at  $x$  such that  $\ell\mathbf{1}_x := 0$  and  $s\mathbf{1}_x := x =: t\mathbf{1}_x$ . For the rest of the paper we assume that the considered quivers do not have oriented cycles, where by an oriented cycle we mean a path  $\sigma$  of positive length such that  $s\sigma = t\sigma$ .

Let  $\Delta$  be a quiver. We define its path category  $\mathbb{k}\Delta$  to be the category whose objects are the vertices of  $\Delta$  and, for  $x, y \in \Delta_0$ , the morphisms from  $x$  to  $y$  are the formal  $\mathbb{k}$ -linear combinations of paths starting at

$x$  and terminating at  $y$ . If  $\omega$  is a morphism from  $x$  to  $y$ , then we write  $s\omega := x$  and  $t\omega := y$ . By a representation of  $\Delta$  we mean a functor from  $\mathbb{k}\Delta$  to the category  $\text{mod } \mathbb{k}$  of finite dimensional vector spaces. We denote the category of representations of  $\Delta$  by  $\text{rep } \Delta$ . Observe that every representation of  $\Delta$  is uniquely determined by its values on the vertices and the arrows. Given a representation  $M$  of  $\Delta$  we denote by  $\mathbf{dim} M$  its dimension vector defined by the formula  $(\mathbf{dim} M)(x) := \dim_{\mathbb{k}} M(x)$ , for  $x \in \Delta_0$ . Observe that  $\mathbf{dim} M \in \mathbb{N}^{\Delta_0}$  for each representation  $M$  of  $\Delta$ . We call the elements of  $\mathbb{N}^{\Delta_0}$  dimension vectors. A dimension vector  $\mathbf{d}$  is called sincere if  $\mathbf{d}(x) \neq 0$  for each  $x \in \Delta_0$ .

By a relation in a quiver  $\Delta$  we mean a  $\mathbb{k}$ -linear combination of paths of lengths at least 2 having a common starting vertex and a common terminating vertex. Note that each relation in a quiver  $\Delta$  is a morphism in  $\mathbb{k}\Delta$ . A set  $\mathfrak{R}$  of relations in a quiver  $\Delta$  is called minimal if  $\langle \mathfrak{R} \setminus \{\rho\} \rangle \neq \langle \mathfrak{R} \rangle$  for each  $\rho \in \mathfrak{R}$ , where for a set  $\mathfrak{X}$  of morphisms in  $\Delta$  we denote by  $\langle \mathfrak{X} \rangle$  the ideal in  $\mathbb{k}\Delta$  generated by  $\mathfrak{X}$ . Observe that each minimal set of relations is finite. By a bound quiver  $\Delta$  we mean a quiver  $\Delta$  together with a minimal set  $\mathfrak{R}$  of relations. Given a bound quiver  $\Delta$  we denote by  $\mathbb{k}\Delta$  its path category, i.e.  $\mathbb{k}\Delta := \mathbb{k}\Delta / \langle \mathfrak{R} \rangle$ . By a representation of a bound quiver  $\Delta$  we mean a functor from  $\mathbb{k}\Delta$  to  $\text{mod } \mathbb{k}$ . In other words, a representation of  $\Delta$  is a representation  $M$  of  $\Delta$  such that  $M(\rho) = 0$  for each  $\rho \in \mathfrak{R}$ . We denote the category of representations of a bound quiver  $\Delta$  by  $\text{rep } \Delta$ . Moreover, we denote by  $\text{ind } \Delta$  the full subcategory of  $\text{rep } \Delta$  consisting of the indecomposable representations. It is known that  $\text{rep } \Delta$  is an abelian Krull–Schmidt category.

An important role in the study of representations of quivers is played by the Auslander–Reiten translations  $\tau$  and  $\tau^-$  [1, Section IV.2], which assign to each representation of a bound quiver  $\Delta$  another representation of  $\Delta$ . In particular, we will use the following consequences of the Auslander–Reiten formulas [1, Theorem IV.2.13]. Let  $M$  and  $N$  be representations of a bound quiver  $\Delta$ . If  $\text{pdim}_{\Delta} M \leq 1$ , then

$$(1.1) \quad \dim_{\mathbb{k}} \text{Ext}_{\Delta}^1(M, N) = \dim_{\mathbb{k}} \text{Hom}_{\Delta}(N, \tau M).$$

Dually, if  $\text{idim}_{\Delta} N \leq 1$ , then

$$(1.2) \quad \dim_{\mathbb{k}} \text{Ext}_{\Delta}^1(M, N) = \dim_{\mathbb{k}} \text{Hom}_{\Delta}(\tau^- N, M).$$

Let  $\Delta$  be a bound quiver. We define the corresponding Tits form  $\langle -, - \rangle_{\Delta} : \mathbb{Z}^{\Delta_0} \times \mathbb{Z}^{\Delta_0} \rightarrow \mathbb{Z}$  by the formula

$$\langle \mathbf{d}', \mathbf{d}'' \rangle_{\Delta} := \sum_{x \in \Delta_0} \mathbf{d}'(x) \cdot \mathbf{d}''(x) - \sum_{\alpha \in \Delta_1} \mathbf{d}'(s\alpha) \cdot \mathbf{d}''(t\alpha) + \sum_{\rho \in \mathfrak{R}} \mathbf{d}'(s\rho) \cdot \mathbf{d}''(t\rho),$$

for  $\mathbf{d}', \mathbf{d}'' \in \mathbb{Z}^{\Delta_0}$ . Bongartz [8, Proposition 2.2] has proved that

$$\begin{aligned} \langle \mathbf{dim} M, \mathbf{dim} N \rangle_{\Delta} \\ = \dim_{\mathbb{k}} \text{Hom}_{\Delta}(M, N) - \dim_{\mathbb{k}} \text{Ext}_{\Delta}^1(M, N) + \dim_{\mathbb{k}} \text{Ext}_{\Delta}^2(M, N) \end{aligned}$$

for any  $M, N \in \text{rep } \Delta$  provided  $\text{gldim } \Delta \leq 2$ .

## 2. SEPARATING EXACT SUBCATEGORIES

In this section we present facts about sincere separating exact subcategories, which we use in our considerations. For the proofs we refer to [23, 26].

Let  $\Delta$  be a bound quiver and  $\mathcal{X}$  a full subcategory of  $\text{ind } \Delta$ . We denote by  $\text{add } \mathcal{X}$  the full subcategory of  $\text{rep } \Delta$  formed by the direct sums of representations from  $\mathcal{X}$ . We say that  $\mathcal{X}$  is an exact subcategory of  $\text{ind } \Delta$  if  $\text{add } \mathcal{X}$  is an exact subcategory of  $\text{rep } \Delta$ , where by an exact subcategory of  $\text{rep } \Delta$  we mean a full subcategory  $\mathcal{E}$  of  $\text{rep } \Delta$  such that  $\mathcal{E}$  is an abelian category and the inclusion functor  $\mathcal{E} \hookrightarrow \text{rep } \Delta$  is exact. We put

$$\mathcal{X}_- := \{X \in \text{ind } \Delta : \text{Hom}_\Delta(\mathcal{X}, X) = 0\}$$

and

$$\mathcal{X}_+ := \{X \in \text{ind } \Delta : \text{Hom}_\Delta(X, \mathcal{X}) = 0\}.$$

Let  $\Delta$  be a bound quiver. Following [23] we say that  $\mathcal{R}$  is a sincere separating exact subcategory of  $\text{ind } \Delta$  provided the following conditions are satisfied:

- (1)  $\mathcal{R}$  is an exact subcategory of  $\text{ind } \Delta$  stable under the actions of the Auslander–Reiten translations  $\tau$  and  $\tau^-$ .
- (2)  $\text{ind } \Delta = \mathcal{R}_- \cup \mathcal{R} \cup \mathcal{R}_+$ .
- (3)  $\text{Hom}_\Delta(X, \mathcal{R}) \neq 0$  for each  $X \in \mathcal{R}_-$  and  $\text{Hom}_\Delta(\mathcal{R}, X) \neq 0$  for each  $X \in \mathcal{R}_+$ .
- (4)  $P \in \mathcal{R}_-$ , for each indecomposable projective representation  $P$  of  $\Delta$ , and  $I \in \mathcal{R}_+$ , for each indecomposable injective representation  $I$  of  $\Delta$ .

Lenzing and de la Peña [23] have proved that there exists a sincere separating exact subcategory  $\mathcal{R}$  of  $\text{ind } \Delta$  if and only if  $\Delta$  is concealed-canonical, i.e.  $\text{rep } \Delta$  is equivalent to the category of modules over a concealed-canonical algebra. In particular, if this is the case, then  $\text{gldim } \Delta \leq 2$ .

For the rest of the section we fix a concealed-canonical bound quiver  $\Delta$  and a sincere separating exact subcategory  $\mathcal{R}$  of  $\text{ind } \Delta$ . Moreover, we put  $\mathcal{P} := \mathcal{R}_-$  and  $\mathcal{Q} := \mathcal{R}_+$ . Finally, we denote by  $\mathbf{P}$ ,  $\mathbf{R}$  and  $\mathbf{Q}$  the dimension vectors of the representations from  $\text{add } \mathcal{P}$ ,  $\text{add } \mathcal{R}$  and  $\text{add } \mathcal{Q}$ , respectively.

It is known that  $\text{pdim}_\Delta P \leq 1$  for each  $P \in \mathcal{P}$  and  $\text{idim}_\Delta Q \leq 1$  for each  $Q \in \mathcal{Q}$ . Next,  $\text{pdim}_\Delta R = 1$  and  $\text{idim}_\Delta R = 1$  for each  $R \in \mathcal{R}$ . The categories  $\mathcal{P}$  and  $\mathcal{Q}$  are closed under the actions of  $\tau$  and  $\tau^-$ , hence using the Auslander–Reiten formulas (1.1) and (1.2) we obtain

that  $\text{Ext}_{\Delta}^1(\mathcal{P}, \mathcal{R}) = 0 = \text{Ext}_{\Delta}^1(\mathcal{R}, \mathcal{Q})$ . In particular,

$$(2.1) \quad \langle \mathbf{d}', \mathbf{d} \rangle_{\Delta} \geq 0 \quad \text{and} \quad \langle \mathbf{d}, \mathbf{d}'' \rangle_{\Delta} \geq 0$$

for all  $\mathbf{d}' \in \mathbf{P}$ ,  $\mathbf{d} \in \mathbf{R}$  and  $\mathbf{d}'' \in \mathbf{Q}$ .

We have  $\mathcal{R} = \coprod_{\lambda \in \mathbb{P}_{\mathbb{k}}^1} \mathcal{R}_{\lambda}$  for connected uniserial categories  $\mathcal{R}_{\lambda}$ ,  $\lambda \in \mathbb{P}_{\mathbb{k}}^1$ . For  $\lambda \in \mathbb{P}_{\mathbb{k}}^1$  we denote by  $r_{\lambda}$  the number of the pairwise non-isomorphic simple objects in  $\text{add } \mathcal{R}_{\lambda}$ . Then  $r_{\lambda} < \infty$  for each  $\lambda \in \mathbb{P}_{\mathbb{k}}^1$ . Moreover,  $\sum_{\lambda \in \mathbb{P}_{\mathbb{k}}^1} (r_{\lambda} - 1) = |\Delta_0| - 2$ . In particular, if  $\mathbb{X}_0 := \{\lambda \in \mathbb{P}_{\mathbb{k}}^1 : r_{\lambda} > 1\}$ , then  $|\mathbb{X}_0| < \infty$ .

Fix  $\lambda \in \mathbb{P}_{\mathbb{k}}^1$ . If  $R_{\lambda,0}, \dots, R_{\lambda,r_{\lambda}-1}$  are chosen representatives of the isomorphism classes of the simple objects in  $\text{add } \mathcal{R}_{\lambda}$ , then we may assume that  $\tau R_{\lambda,i} = R_{\lambda,i-1}$  for each  $i \in [0, r_{\lambda}-1]$ , where we put  $R_{\lambda,i} := R_{\lambda,i \bmod r_{\lambda}}$ , for  $i \in \mathbb{Z}$ . For any  $i \in \mathbb{Z}$  and  $n \in \mathbb{N}_+$  there exists a unique (up to isomorphism) representation in  $\mathcal{R}_{\lambda}$  whose socle and length in  $\text{add } \mathcal{R}_{\lambda}$  are  $R_{\lambda,i}$  and  $n$ , respectively. We fix such representation and denote it by  $R_{\lambda,i}^{(n)}$  and its dimension vector by  $\mathbf{e}_{\lambda,i}^n$ . Then the composition factors of  $R_{\lambda,i}^{(n)}$  are (starting from the socle)  $R_{\lambda,i}, \dots, R_{\lambda,i+n-1}$ . Consequently,  $\mathbf{e}_{\lambda,i}^n = \sum_{j \in [i, i+n-1]} \mathbf{e}_{\lambda,j}$ , where  $\mathbf{e}_{\lambda,j} := \mathbf{dim } R_{\lambda,j}$ , for  $j \in \mathbb{Z}$ . Moreover, for all  $i \in \mathbb{Z}$  and  $n, m \in \mathbb{N}_+$  there exists an exact sequence

$$(2.2) \quad 0 \rightarrow R_{\lambda,i}^{(n)} \rightarrow R_{\lambda,i}^{(n+m)} \rightarrow R_{\lambda,i+n}^{(m)} \rightarrow 0.$$

Obviously, for each  $R \in \mathcal{R}_{\lambda}$  there exist  $i \in \mathbb{Z}$  and  $n \in \mathbb{N}_+$  such that  $R \simeq R_{\lambda,i}^{(n)}$ . Moreover, it is known that the vectors  $\mathbf{e}_{\lambda,0}, \dots, \mathbf{e}_{\lambda,r_{\lambda}-1}$  are linearly independent. Consequently, if  $R \in \text{add } \mathcal{R}_{\lambda}$ , then there exist uniquely determined  $q_0^R, \dots, q_{r_{\lambda}-1}^R \in \mathbb{N}$  such that  $\mathbf{dim } R = \sum_{i \in [0, r_{\lambda}-1]} q_i^R \mathbf{e}_{\lambda,i}$ . We put  $q_i^R := q_{i \bmod r_{\lambda}}^R$ , for  $i \in \mathbb{Z}$ . Observe that for each  $i \in \mathbb{Z}$  the number  $q_{\lambda,i}^R$  counts the multiplicity of  $R_{\lambda,i}$  as a composition factor in the Jordan–Hölder filtration of  $R$  in the category  $\text{add } \mathcal{R}_{\lambda}$ .

Let  $R = \bigoplus_{\lambda \in \mathbb{P}_{\mathbb{k}}^1} R_{\lambda}$ , for  $R_{\lambda} \in \text{add } \mathcal{R}_{\lambda}$ ,  $\lambda \in \mathbb{P}_{\mathbb{k}}^1$ . Then we put  $q_{\lambda,i}^R := q_i^{R_{\lambda}}$  for  $\lambda \in \mathbb{P}_{\mathbb{k}}^1$  and  $i \in \mathbb{Z}$ . Next, we put  $p_{\lambda}^R := \min\{q_{\lambda,i}^R : i \in \mathbb{Z}\}$ , for  $\lambda \in \mathbb{P}_{\mathbb{k}}^1$ , and  $p_{\lambda,i}^R := q_{\lambda,i}^R - p_{\lambda}^R$ , for  $\lambda \in \mathbb{P}_{\mathbb{k}}^1$  and  $i \in \mathbb{Z}$ . Then

$$\mathbf{dim } R = \sum_{\lambda \in \mathbb{P}_{\mathbb{k}}^1} p_{\lambda}^R \cdot \mathbf{h}_{\lambda} + \sum_{\lambda \in \mathbb{P}_{\mathbb{k}}^1} \sum_{i \in [0, r_{\lambda}-1]} p_{\lambda,i}^R \cdot \mathbf{e}_{\lambda,i},$$

where  $\mathbf{h}_{\lambda} := \sum_{i \in [0, r_{\lambda}-1]} \mathbf{e}_{\lambda,i}$ , for  $\lambda \in \mathbb{P}_{\mathbb{k}}^1$ . It is known that  $\mathbf{h}_{\lambda} = \mathbf{h}_{\mu}$  for any  $\lambda, \mu \in \mathbb{P}_{\mathbb{k}}^1$ . We denote this common value by  $\mathbf{h}$ . Then

$$\mathbf{dim } R = p^R \cdot \mathbf{h} + \sum_{\lambda \in \mathbb{P}_{\mathbb{k}}^1} \sum_{i \in [0, r_{\lambda}-1]} p_{\lambda,i}^R \cdot \mathbf{e}_{\lambda,i},$$

where  $p^R := \sum_{\lambda \in \mathbb{P}_{\mathbb{k}}^1} p_{\lambda}^R$ . It is known that if  $R, R' \in \text{add } \mathcal{R}$  and  $\mathbf{dim } R = \mathbf{dim } R'$ , then  $p^R = p^{R'}$  and  $p_{\lambda,i}^R = p_{\lambda,i}^{R'}$  for any  $\lambda \in \mathbb{P}_{\mathbb{k}}^1$  and  $i \in [0, r_{\lambda}-1]$ .

Consequently, for each  $\mathbf{d} \in \mathbf{R}$  there exist uniquely determined  $p^{\mathbf{d}} \in \mathbb{N}$  and  $p_{\lambda,i}^{\mathbf{d}} \in \mathbb{N}$  for  $\lambda \in \mathbb{P}_{\mathbb{k}}^1$  and  $i \in [0, r_{\lambda} - 1]$ , such that

$$\mathbf{d} = p^{\mathbf{d}} \cdot \mathbf{h} + \sum_{\lambda \in \mathbb{P}_{\mathbb{k}}^1} \sum_{i \in [0, r_{\lambda} - 1]} p_{\lambda,i}^{\mathbf{d}} \cdot \mathbf{e}_{\lambda,i}$$

and for each  $\lambda \in \mathbb{P}_{\mathbb{k}}^1$  there exists  $i \in [0, r_{\lambda} - 1]$  with  $p_{\lambda,i}^{\mathbf{d}} = 0$ . Again we put  $p_{\lambda,i}^{\mathbf{d}} := p_{\lambda, i \bmod r_{\lambda}}^{\mathbf{d}}$ , for  $\mathbf{d} \in \mathbf{R}$ ,  $\lambda \in \mathbb{P}_{\mathbb{k}}^1$  and  $i \in \mathbb{Z}$ .

It is known that  $\mathbf{h}$  is sincere. Moreover,  $\mathbf{h}$  can be used in order to distinguish between representations from  $\mathcal{P}$ ,  $\mathcal{Q}$  and  $\mathcal{R}$ . Namely, if  $X$  is an indecomposable representation of  $\Delta$ , then

$$(2.3) \quad X \in \mathcal{P} \quad \text{if and only if} \quad \langle \mathbf{dim} X, \mathbf{h} \rangle_{\Delta} > 0.$$

Dually, if  $X$  is an indecomposable representation of  $\Delta$ , then

$$(2.4) \quad X \in \mathcal{Q} \quad \text{if and only if} \quad \langle \mathbf{h}, \mathbf{dim} X \rangle_{\Delta} > 0.$$

Let  $\lambda, \mu \in \mathbb{P}_{\mathbb{k}}^1$ ,  $i, j \in \mathbb{Z}$  and  $m, n \in \mathbb{N}_+$ . Then

$$\dim_{\mathbb{k}} \text{Hom}_{\Delta}(R_{\lambda,i}^{(n)}, R_{\mu,j}^{(m)}) = \min\{q_{\lambda, i+n-1}^{R_{\mu,j}^{(m)}}, q_{\mu,j}^{R_{\lambda,i}^{(n)}}\}$$

(in particular,  $\text{Hom}_{\Delta}(R_{\lambda,i}^{(n)}, R_{\mu,j}^{(m)}) = 0$  if  $\lambda \neq \mu$ ). The above formula, together with the Auslander–Reiten formula (1.1), implies that

$$(2.5) \quad \langle \mathbf{e}_{\lambda,i}^n, \mathbf{d} \rangle_{\Delta} = p_{\lambda, i+n-1}^{\mathbf{d}} - p_{\lambda, i-1}^{\mathbf{d}}$$

for any  $\lambda \in \mathbb{P}_{\mathbb{k}}^1$ ,  $i \in \mathbb{Z}$ ,  $n \in \mathbb{N}_+$  and  $\mathbf{d} \in \mathbf{R}$ . In particular,

$$(2.6) \quad \langle \mathbf{h}, \mathbf{d} \rangle_{\Delta} = 0 = \langle \mathbf{d}, \mathbf{h} \rangle_{\Delta}$$

for each  $\mathbf{d} \in \mathbf{R}$ .

An important role in the proofs will be played by ext-minimal representations. We call a representation  $V$  ext-minimal if there is no decomposition  $V = V_1 \oplus V_2$  with  $\text{Ext}_{\Delta}^1(V_1, V_2) \neq 0$ . We recall facts on ext-minimal representations belonging to  $\text{add } \mathcal{R}$ .

First assume that  $\mathbf{d} \in \mathbf{R}$  and  $p^{\mathbf{d}} = 0$ . In this case there is a unique (up to isomorphism) ext-minimal representation  $W \in \text{add } \mathcal{R}$  with dimension vector  $\mathbf{d}$ , which is constructed inductively in the following way. For  $\lambda \in \mathbb{P}_{\mathbb{k}}^1$ , let  $I_{\lambda} := \{i \in [0, r_{\lambda} - 1] : p_{\lambda,i}^{\mathbf{d}} \neq 0 \text{ and } p_{\lambda, i-1}^{\mathbf{d}} = 0\}$ . For  $\lambda \in \mathbb{P}_{\mathbb{k}}^1$  and  $i \in I_{\lambda}$ , we denote by  $m_{\lambda,i}$  the minimal  $m \in \mathbb{N}_+$  such that  $p_{\lambda, i+m}^{\mathbf{d}} = 0$ . By induction there exists (unique up to isomorphism) ext-minimal representation  $W' \in \text{add } \mathcal{R}$  with dimension vector  $\mathbf{d} - \sum_{\lambda \in \mathbb{P}_{\mathbb{k}}^1} \sum_{i \in I_{\lambda}} \mathbf{e}_{\lambda,i}^{m_{\lambda,i}}$ . Then  $W := W' \oplus \bigoplus_{\lambda \in \mathbb{P}_{\mathbb{k}}^1} \bigoplus_{i \in I_{\lambda}} R_{\lambda,i}^{(m_{\lambda,i})}$  is ext-minimal.

We will use the following property of the above representation.

**Lemma 2.1.** *Assume  $\mathbf{d} \in \mathbf{R}$  and  $p^{\mathbf{d}} = 0$ . Let  $W \in \text{add } \mathcal{R}$  be an ext-minimal representation with dimension vector  $\mathbf{d}$ . If  $\lambda \in \mathbb{P}_{\mathbb{k}}^1$ ,  $i \in \mathbb{Z}$ ,  $n \in \mathbb{N}_+$ ,  $p_{\lambda,i}^{\mathbf{d}} = p_{\lambda, i+n}^{\mathbf{d}}$  and  $p_{\lambda,j}^{\mathbf{d}} \geq p_{\lambda,i}^{\mathbf{d}}$  for each  $j \in [i, i+n]$ , then  $\text{Hom}_{\Delta}(R_{\lambda, i+1}^{(n)}, W) = 0$ .*

*Proof.* Observe that  $\text{Hom}_{\Delta}(R_{\lambda,i+1}^{(n)}, R_{\lambda,k}^{(m_{\lambda,k})}) = 0$  for each  $k \in I_{\lambda}$ , since one easily checks that either  $q_{\lambda,i+n}^{R_{\lambda,k}^{(m_{\lambda,k})}} = 0$  (if  $p_{\lambda,i}^{\mathbf{d}} = 0$ ) or  $q_{\lambda,k}^{R_{\lambda,i+1}^{(n)}} = 0$  (if  $p_{\lambda,i}^{\mathbf{d}} > 0$ ). Now the claim follows by induction.  $\square$

Now let  $\mathbf{d} \in \mathbf{R}$  be arbitrary. The description of the ext-minimal representations with dimension vector  $\mathbf{d}$ , which belong to  $\text{add } \mathcal{R}$ , has been given in [25, Theorem 3.5] (this theorem has been formulated in the case  $\Delta = (\Delta, \emptyset)$  for a Euclidean quiver  $\Delta$ , but its proof translates to an arbitrary concealed-canonical bound quiver). We will not repeat the formulation here, but only mention some consequences. First, if  $W \in \text{add } \mathcal{R}$  and  $\dim W = \mathbf{d}$ , then  $W$  is ext-minimal if and only if  $\dim_{\mathbb{k}} \text{End}_{\Delta}(W) = p^{\mathbf{d}} + \langle \mathbf{d}, \mathbf{d} \rangle_{\Delta}$ . In particular,

$$(2.7) \quad p^{\mathbf{d}} + \langle \mathbf{d}, \mathbf{d} \rangle_{\Delta} = \min\{\dim_{\mathbb{k}} \text{End}_{\Delta}(W) : W \in \text{add } \mathcal{R} \text{ such that } \dim W = \mathbf{d}\}$$

(here we use also [25, Lemma 2.1]). Next, if  $W \in \text{add } \mathcal{R}$  is an ext-minimal representation with dimension vector  $\mathbf{d}$  and  $W' \in \text{add } \mathcal{R}$  is an ext-minimal representation with dimension vector  $\mathbf{d} - p^{\mathbf{d}} \cdot \mathbf{h}$ , then there exists an exact sequence  $0 \rightarrow \bigoplus_{\lambda \in \mathbb{P}_{\mathbb{k}}^1} R_{\lambda} \rightarrow W \rightarrow W' \rightarrow 0$  with  $R_{\lambda} \in \mathcal{R}_{\lambda}$  (in particular, indecomposable) for each  $\lambda \in \mathbb{P}_{\mathbb{k}}^1$  (obviously,  $\dim R_{\lambda}$  is a multiplicity of  $\mathbf{h}$  for each  $\lambda \in \mathbb{P}_{\mathbb{k}}^1$ ).

### 3. SEMI-INVARIANTS

Let  $\Delta$  be a bound quiver and  $\mathbf{d}$  a dimension vector. By  $\text{rep}_{\Delta}(\mathbf{d})$  we denote the set of the representations  $M$  of  $\Delta$  such that  $M(x) = \mathbb{k}^{\mathbf{d}(x)}$  for each  $x \in \Delta_0$ . We may identify  $\text{rep}_{\Delta}(\mathbf{d})$  with a Zariski-closed subset of the affine space  $\text{rep}_{\Delta}(\mathbf{d}) := \prod_{\alpha \in \Delta_1} \mathbb{M}_{\mathbf{d}(t\alpha) \times \mathbf{d}(s\alpha)}(\mathbb{k})$ , hence it has a structure of an affine variety. The group  $\text{GL}(\mathbf{d}) := \prod_{x \in \Delta_0} \text{GL}(\mathbf{d}(x))$  acts on  $\text{rep}_{\Delta}(\mathbf{d})$  by conjugation:  $(g * M)(\alpha) := g(t\alpha) \cdot M(\alpha) \cdot g(s\alpha)^{-1}$ , for  $g \in \text{GL}(\mathbf{d})$ ,  $M \in \text{rep}_{\Delta}(\mathbf{d})$  and  $\alpha \in \Delta_1$ . The set  $\text{rep}_{\Delta}(\mathbf{d})$  is a  $\text{GL}(\mathbf{d})$ -invariant subset of  $\text{rep}_{\Delta}(\mathbf{d})$  and the  $\text{GL}(\mathbf{d})$ -orbits in  $\text{rep}_{\Delta}(\mathbf{d})$  correspond to the isomorphism classes of the representations of  $\Delta$  with dimension vector  $\mathbf{d}$ . If  $\mathcal{X}$  is a full subcategory of  $\text{ind } \Delta$ , then we denote by  $\mathcal{X}(\mathbf{d})$  the set of  $V \in \text{rep}_{\Delta}(\mathbf{d})$  such that  $V \in \text{add } \mathcal{X}$ .

Let  $\Delta$  be a quiver and  $\theta \in \mathbb{Z}^{\Delta_0}$ . We treat  $\theta$  as a  $\mathbb{Z}$ -linear function  $\mathbb{Z}^{\Delta_0} \rightarrow \mathbb{Z}$  in a usual way. If  $\mathbf{d}$  is a dimension vector, then by a semi-invariant of weight  $\theta$  we mean every function  $f \in \mathbb{k}[\text{rep}_{\Delta}(\mathbf{d})]$  such that  $f(g^{-1} * M) = \chi^{\theta}(g) \cdot f(M)$  for any  $g \in \text{GL}(\mathbf{d})$  and  $M \in \text{rep}_{\Delta}(\mathbf{d})$ , where  $\chi^{\theta}(g) := \prod_{x \in \Delta_0} (\det g(x))^{\theta(x)}$  for  $g \in \text{GL}(\mathbf{d})$ .

Now let  $\Delta$  be a bound quiver and  $\mathbf{d}$  a dimension vector. If  $\theta \in \mathbb{Z}^{\Delta_0}$ , then a function  $f \in \mathbb{k}[\text{rep}_{\Delta}(\mathbf{d})]$  is called a semi-invariant of weight  $\theta$  if  $f$  is the restriction of a semi-invariant of weight  $\theta$  from  $\mathbb{k}[\text{rep}_{\Delta}(\mathbf{d})]$ . This definition differs from the definition used in other papers on the subject

(see for example [5, 11, 13, 15]), however these are the semi-invariants which one needs to understand in order to study King's moduli spaces for representations of bound quivers [21]. Moreover, the two definitions coincide if the characteristic of  $\mathbb{k}$  equals 0. We denote the space of the semi-invariants of weight  $\theta$  by  $\text{SI}[\Delta, \mathbf{d}]_\theta$ . If  $\mathbf{d}$  is sincere, then we put  $\text{SI}[\Delta, \mathbf{d}] := \bigoplus_{\theta \in \mathbb{Z}^{\Delta_0}} \text{SI}[\Delta, \mathbf{d}]_\theta$  and call it the algebra of semi-invariants for  $\Delta$  and  $\mathbf{d}$  (we assume sincerity of  $\mathbf{d}$ , since under this assumption  $\mathbb{Z}^{\Delta_0}$  is isomorphic with the character group of  $\text{GL}(\mathbf{d})$ ).

We recall a construction from [13]. Let  $\Delta$  be a bound quiver. Fix a representation  $V$  of  $\Delta$  and define  $\theta^V : \mathbb{Z}^{\Delta_0} \rightarrow \mathbb{Z}$  by the condition:

$$\theta^V(\dim M) = \dim_{\mathbb{k}} \text{Hom}_{\Delta}(V, M) - \dim_{\mathbb{k}} \text{Hom}_{\Delta}(M, \tau V)$$

for each representation  $M$  of  $\Delta$ . The formula (1.1) implies that  $\theta^V = \langle \dim V, - \rangle_{\Delta}$  if  $\text{pdim}_{\Delta} V \leq 1$ . Dually, if  $V$  has no indecomposable projective direct summands (i.e.  $\tau^- \tau V \simeq V$  [1, Theorem IV.2.10]) and  $\text{idim}_{\Delta} \tau V \leq 1$ , then  $\theta^V = -\langle -, \dim \tau V \rangle_{\Delta}$  by the formula (1.2).

Now let  $\mathbf{d}$  be a dimension vector. If  $\theta^V(\mathbf{d}) = 0$ , then we define a function  $c_{\mathbf{d}}^V \in \mathbb{k}[\text{rep}_{\Delta}(\mathbf{d})]$  in the following way. Let  $P_1 \xrightarrow{f} P_0 \rightarrow V \rightarrow 0$  be the minimal projective presentation of  $V$ . One shows that

$$\dim_{\mathbb{k}} \text{Ker Hom}_{\Delta}(f, M) = \dim_{\mathbb{k}} \text{Hom}_{\Delta}(V, M)$$

and

$$\dim_{\mathbb{k}} \text{Coker Hom}_{\Delta}(f, M) = \dim_{\mathbb{k}} \text{Hom}_{\Delta}(M, \tau V),$$

hence

$$\begin{aligned} (3.1) \quad & \dim_{\mathbb{k}} \text{Hom}_{\Delta}(P_0, M) - \dim_{\mathbb{k}} \text{Hom}_{\Delta}(P_1, M) \\ &= \dim_{\mathbb{k}} \text{Hom}_{\Delta}(V, M) - \dim_{\mathbb{k}} \text{Hom}_{\Delta}(M, \tau V) = \theta^V(\mathbf{d}) = 0, \end{aligned}$$

for each  $M \in \text{rep}_{\Delta}(\mathbf{d})$ . Thus, we may define  $c_{\mathbf{d}}^V \in \mathbb{k}[\text{rep}_{\Delta}(\mathbf{d})]$  by the formula  $c_{\mathbf{d}}^V(M) := \det \text{Hom}_{\Delta}(f, M)$  for  $M \in \text{rep}_{\Delta}(\mathbf{d})$ . Note that  $c_{\mathbf{d}}^V$  is defined only up to a non-zero scalar. If  $M \in \text{rep}_{\Delta}(\mathbf{d})$ , then  $c_{\mathbf{d}}^V(M) = 0$  if and only if  $\text{Hom}_{\Delta}(V, M) \neq 0$ . Moreover, if  $\text{pdim}_{\Delta} V \leq 1$  and  $M \in \text{rep}_{\Delta}(\mathbf{d})$ , then  $c_{\mathbf{d}}^V(M) = 0$  if and only if  $\text{Ext}_{\Delta}^1(V, M) \neq 0$ . It is known that  $c_{\mathbf{d}}^V \in \text{SI}[\Delta, \mathbf{d}]_{\theta^V}$ . This function depends on the choice of  $f$ , but the functions obtained for different  $f$ 's differ only by non-zero scalars.

In fact, we could start with an arbitrary  $\mathbf{d}$ -admissible projective presentation, where, for a representation  $V$  of a bound quiver  $\Delta$  and a dimension vector  $\mathbf{d}$ , we call a projective representation  $P'_1 \rightarrow P'_0 \rightarrow V \rightarrow 0$  of  $V$   $\mathbf{d}$ -admissible if  $\dim_{\mathbb{k}} \text{Hom}_{\Delta}(P'_0, M) = \dim_{\mathbb{k}} \text{Hom}_{\Delta}(P'_1, M)$  for any (equivalently, some)  $M \in \text{rep}_{\Delta}(\mathbf{d})$ .

**Lemma 3.1.** *Let  $\Delta$  be a bound quiver,  $\mathbf{d}$  a dimension vector and  $P'_1 \xrightarrow{f'} P'_0 \rightarrow V \rightarrow 0$  a  $\mathbf{d}$ -admissible projective presentation of a representation  $V$  of  $\Delta$ .*



- (1) If  $\theta^V(\mathbf{d}) = 0$ , then there exists  $\xi \in \mathbb{k}$  such  $\xi \neq 0$  and  $c_{\mathbf{d}}^V(M) = \xi \cdot \det \operatorname{Hom}_{\Delta}(f', M)$  for each  $M \in \operatorname{rep}_{\Delta}(\mathbf{d})$ .
- (2) If there exists  $M \in \operatorname{rep}_{\Delta}(\mathbf{d})$  such that  $\det \operatorname{Hom}_{\Delta}(f', M) \neq 0$ , then  $\theta^V(\mathbf{d}) = 0$ .

*Proof.* Let  $P_1 \xrightarrow{f} P_0 \rightarrow V \rightarrow 0$  be the minimal projective presentation of  $V$ . There exists projective representations  $P$  and  $Q$  of  $\Delta$  and isomorphisms  $g_1 : P'_1 \rightarrow P_1 \oplus P \oplus Q$  and  $g_0 : P'_0 \rightarrow P_0 \oplus P$  such that

$$f' = g_0^{-1} \circ \begin{bmatrix} f & 0 & 0 \\ 0 & \operatorname{Id}_P & 0 \end{bmatrix} \circ g_1.$$

Consequently,

$$(3.2) \quad \operatorname{Hom}_{\Delta}(f', M) = \operatorname{Hom}_{\Delta}(g_1, M) \circ \begin{bmatrix} \operatorname{Hom}_{\Delta}(f, M) & 0 \\ 0 & \operatorname{Hom}_{\Delta}(\operatorname{Id}_P, M) \\ 0 & 0 \end{bmatrix} \circ \operatorname{Hom}_{\Delta}(g_0^{-1}, M)$$

for each  $M \in \operatorname{rep}_{\Delta}(\mathbf{d})$ . Since the presentation  $P'_1 \xrightarrow{f'} P'_0 \rightarrow V \rightarrow 0$  is  $\mathbf{d}$ -admissible, (3.1) implies that the condition  $\theta^V(\mathbf{d}) = 0$  is equivalent to the condition  $\dim_{\mathbb{k}} \operatorname{Hom}_{\Delta}(Q, M) = 0$  for each  $M \in \operatorname{rep}_{\Delta}(\mathbf{d})$ . Together with (3.2) this implies our claims.  $\square$

As an immediate consequence we obtain the following.

**Corollary 3.2.** *Let  $\Delta$  be a bound quiver,  $\mathbf{d}$  a dimension vector and  $0 \rightarrow V_1 \rightarrow V \rightarrow V_2 \rightarrow 0$  an exact sequence such that  $\theta^{V_1}(\mathbf{d}) = 0 = \theta^{V_2}(\mathbf{d})$ .*

- (1) *If  $\theta^V(\mathbf{d}) = 0$ , then (up to a non-zero scalar)  $c_{\mathbf{d}}^V = c_{\mathbf{d}}^{V_1} \cdot c_{\mathbf{d}}^{V_2}$ .*
- (2) *If  $c_{\mathbf{d}}^{V_1} \cdot c_{\mathbf{d}}^{V_2} \neq 0$ , then  $\theta^V(\mathbf{d}) = 0$  and (up to a non-zero scalar)  $c_{\mathbf{d}}^V = c_{\mathbf{d}}^{V_1} \cdot c_{\mathbf{d}}^{V_2}$ .*

*Proof.* Let  $P'_1 \xrightarrow{f'} P'_0 \rightarrow V_1 \rightarrow 0$  and  $P''_1 \xrightarrow{f''} P'_0 \rightarrow V_2 \rightarrow 0$  be the minimal projective presentations of  $V_1$  and  $V_2$ , respectively. Then there exists a projective presentation of  $V$  of the form

$$P'_1 \oplus P''_1 \xrightarrow{f} P'_0 \oplus P''_0 \rightarrow V \rightarrow 0,$$

where  $f = \begin{bmatrix} f' & g \\ 0 & f'' \end{bmatrix}$  for some  $g \in \operatorname{Hom}_{\Delta}(P''_1, P'_0)$ . One easily sees that  $\det \operatorname{Hom}_{\Delta}(f, M) = c_{\mathbf{d}}^{V_1}(M) \cdot c_{\mathbf{d}}^{V_2}(M)$  for each  $M \in \operatorname{rep}_{\Delta}(\mathbf{d})$ , hence the claims follows from Lemma 3.1.  $\square$

The following fact is an extension of [10, Lemma 1(a)] to the setup of bound quivers.

**Lemma 3.3.** *Let  $\Delta$  be a bound quiver and  $\mathbf{d}$  a dimension vector. If  $0 \rightarrow V_1 \rightarrow V \rightarrow V_2 \rightarrow 0$  is an exact sequence,  $\theta^V(\mathbf{d}) = 0$  and  $c_{\mathbf{d}}^V \neq 0$ , then  $\theta^{V_2}(\mathbf{d}) \leq 0$ .*

*Proof.* If  $\theta^{V_2}(\mathbf{d}) > 0$ , then

$$\dim_{\mathbb{k}} \operatorname{Hom}_{\Delta}(V_2, M) \geq \theta^{V_2}(\mathbf{d}) > 0$$

for each  $M \in \operatorname{rep}_{\Delta}(\mathbf{d})$ . This immediately implies that  $\operatorname{Hom}_{\Delta}(V, M) \neq 0$  for each  $M \in \operatorname{rep}_{\Delta}(\mathbf{d})$ , hence  $c_{\mathbf{d}}^V = 0$ , contradiction.  $\square$

We have the following multiplicative property.

**Lemma 3.4.** *Let  $\Delta$  be a bound quiver and  $\mathbf{d}$  a dimension vector. If  $V_1$  and  $V_2$  are representations of  $\Delta$ ,  $V := V_1 \oplus V_2$ ,  $\theta^V(\mathbf{d}) = 0$  and  $c_{\mathbf{d}}^V \neq 0$ , then  $\theta^{V_1}(\mathbf{d}) = 0 = \theta^{V_2}(\mathbf{d})$  and  $c_{\mathbf{d}}^V = c_{\mathbf{d}}^{V_1} \cdot c_{\mathbf{d}}^{V_2}$  (up to a non-zero scalar).*

*Proof.* See [13, Lemma 3.3].  $\square$

We will also use another multiplicative property.

**Lemma 3.5.** *Let  $\Delta$  be a bound quiver and  $V$  a representation of  $\Delta$ . If  $\mathbf{d}'$  and  $\mathbf{d}''$  are dimension vectors and  $\theta^V(\mathbf{d}') = 0 = \theta^V(\mathbf{d}'')$ , then*

$$c_{\mathbf{d}'+\mathbf{d}''}^V(W' \oplus W'') = c_{\mathbf{d}'}^V(W') \cdot c_{\mathbf{d}''}^V(W'')$$

for all  $(W', W'') \in \operatorname{rep}_{\Delta}(\mathbf{d}') \times \operatorname{rep}_{\Delta}(\mathbf{d}'')$ .

*Proof.* Let  $P_1 \xrightarrow{f} P_0 \rightarrow V \rightarrow 0$  be the minimal projective presentation of  $V$ . If  $(W', W'') \in \operatorname{rep}_{\Delta}(\mathbf{d}') \times \operatorname{rep}_{\Delta}(\mathbf{d}'')$ , then

$$\operatorname{Hom}_{\Delta}(f, W' \oplus W'') = \begin{bmatrix} \operatorname{Hom}_{\Delta}(f, W') & 0 \\ 0 & \operatorname{Hom}_{\Delta}(f, W'') \end{bmatrix}$$

and both  $\operatorname{Hom}_{\Delta}(f, W')$  and  $\operatorname{Hom}_{\Delta}(f, W'')$  are square matrices, hence the claim follows.  $\square$

The following result follows from the proof of [13, Theorem 3.2] (note that the assumption about the characteristic of  $\mathbb{k}$  made in [13, Theorem 3.2] is only necessary to prove surjectivity of the restriction morphism, which we have for free with our definition of semi-invariants).

**Proposition 3.6.** *Let  $\Delta$  be a bound quiver,  $\mathbf{d}$  a dimension vector and  $\theta \in \mathbb{Z}^{\Delta_0}$ .*

- (1) *If  $\theta(\mathbf{d}) \neq 0$ , then  $\operatorname{SI}[\Delta, \mathbf{d}]_{\theta} = 0$ .*
- (2) *If  $\theta(\mathbf{d}) = 0$ , then the space  $\operatorname{SI}[\Delta, \mathbf{d}]_{\theta}$  is spanned by the functions  $c_{\mathbf{d}}^V$  for  $V \in \operatorname{rep} \Delta$  such that  $\theta^V = \theta$  and  $c_{\mathbf{d}}^V \neq 0$ .*  $\square$

In fact we may take a smaller spanning set.

**Corollary 3.7.** *Let  $\Delta$  be a bound quiver and  $\mathbf{d}$  a dimension vector. If  $\theta \in \mathbb{Z}^{\Delta_0}$  and  $\theta(\mathbf{d}) = 0$ , then the space  $\operatorname{SI}[\Delta, \mathbf{d}]_{\theta}$  is spanned by the functions  $c_{\mathbf{d}}^V$  for ext-minimal  $V \in \operatorname{rep} \Delta$  such that  $\theta^V = \theta$  and  $c_{\mathbf{d}}^V \neq 0$ .*

*Proof.* Assume that  $V$  is a representation of  $\Delta$  such that  $\theta^V = \theta$ ,  $c_{\mathbf{d}}^V \neq 0$  and there is a decomposition  $V = V_1 \oplus V_2$  with  $\operatorname{Ext}_{\Delta}^1(V_1, V_2) \neq 0$ . Lemma 3.4 implies that  $\theta^{V_1}(\mathbf{d}) = 0 = \theta^{V_2}(\mathbf{d})$  and  $c_{\mathbf{d}}^{V_1} \cdot c_{\mathbf{d}}^{V_2} \neq 0$ . If  $0 \rightarrow V_2 \rightarrow W \rightarrow V_1 \rightarrow 0$  is a non-split exact sequence, then

Corollary 3.2(2) and Lemma 3.4 imply that (up to a non-zero scalar)  $c_{\mathbf{d}}^W = c_{\mathbf{d}}^{V_1} \cdot c_{\mathbf{d}}^{V_2} = c_{\mathbf{d}}^V$ . Since  $\dim_{\mathbb{k}} \text{End}_{\Delta}(W) < \dim_{\mathbb{k}} \text{End}_{\Delta}(V)$  (see for example [25, Lemma 2.1]), the claim follows by induction.  $\square$

We may even take a smaller set, if we are only interested in generators of  $\text{SI}[\Delta, \mathbf{d}]$ . Namely, we have the following.

**Corollary 3.8.** *Let  $\Delta$  be a bound quiver and  $\mathbf{d}$  a sincere dimension vector. Then the algebra  $\text{SI}[\Delta, \mathbf{d}]$  is generated by the semi-invariants  $c_{\mathbf{d}}^V$  for  $V \in \text{rep}_{\Delta}(\mathbf{d})$  such that  $\theta^V(\mathbf{d}) = 0$ ,  $c_{\mathbf{d}}^V \neq 0$  and  $V$  is indecomposable.*

*Proof.* This follows from Proposition 3.6 and Lemma 3.4 (this is also the content of [13, Corollary 3.4]).  $\square$

#### 4. PRELIMINARY RESULTS

Throughout this section we fix a concealed-canonical bound quiver  $\Delta$  and a sincere separating exact subcategory  $\mathcal{R}$  of  $\text{ind } \Delta$ . We will use notation introduced in Section 2. We also fix  $\mathbf{d} \in \mathbf{R}$  such that  $p := p^{\mathbf{d}} > 0$ . Notice that this implies that  $\mathbf{d}$  is sincere.

First we prove that the algebra  $\text{SI}[\Delta, \mathbf{d}]$  is controlled by the representations from  $\text{add } \mathcal{R}$ .

**Lemma 4.1.** *Let  $V$  be a representation of  $\Delta$  such that  $\theta^V(\mathbf{d}) = 0$ . If  $c_{\mathbf{d}}^V \neq 0$ , then  $V \in \text{add } \mathcal{R}$  and  $\theta^V = \langle \mathbf{dim } V, - \rangle_{\Delta}$ .*

*Proof.* Assume that  $P \in \mathcal{P}$  is a direct summand of  $V$ . Since  $\text{pdim}_{\Delta} P \leq 1$ , (2.1) and (2.3) imply that

$$\theta^P(\mathbf{d}) = \langle \mathbf{dim } P, \mathbf{d} \rangle_{\Delta} \geq \langle \mathbf{dim } P, \mathbf{h} \rangle_{\Delta} > 0.$$

Consequently,  $c_{\mathbf{d}}^V = 0$  by Lemma 3.4, contradiction. Dually,  $V$  cannot have a direct summand from  $\mathcal{Q}$ . Finally, since  $\text{pdim}_{\Delta} V = 1$ ,  $\theta^V = \langle \mathbf{dim } V, - \rangle_{\Delta}$ .  $\square$

Together with Corollary 3.7 this lemma immediately implies the following.

**Corollary 4.2.** *Let  $\theta \in \mathbb{Z}^{\Delta_0}$  be such that  $\text{SI}[\Delta, \mathbf{d}]_{\theta} \neq 0$ . Then there exists  $\mathbf{r} \in \mathbf{R}$  such that  $\theta = \langle \mathbf{r}, - \rangle_{\Delta}$  and  $\langle \mathbf{r}, \mathbf{d} \rangle_{\Delta} = 0$ .*  $\square$

Taking into account Corollary 3.8 we need to identify  $V \in \text{ind } \Delta$  such that  $\theta^V(\mathbf{d}) = 0$  and  $c_{\mathbf{d}}^V \neq 0$ . The first step in this direction is the following.

**Lemma 4.3.** *Let  $V$  be an indecomposable representation of  $\Delta$ . If  $\theta^V(\mathbf{d}) = 0$  and  $c_{\mathbf{d}}^V \neq 0$ , then  $V = R_{\lambda, i+1}^{(n)}$  for some  $\lambda \in \mathbb{P}_{\mathbb{k}}^1$ ,  $i \in \mathbb{Z}$  and  $n \in \mathbb{N}_+$  such that  $p_{\lambda, i}^{\mathbf{d}} = p_{\lambda, i+n}^{\mathbf{d}}$  and  $p_{\lambda, j}^{\mathbf{d}} \geq p_{\lambda, i}^{\mathbf{d}}$  for each  $j \in [i+1, i+n-1]$ .*

*Proof.* We know from Lemma 4.1 that  $V \in \mathcal{R}$ , hence there exists  $\lambda \in \mathbb{P}_{\mathbb{k}}^1$ ,  $i \in \mathbb{Z}$  and  $n \in \mathbb{N}_+$  such that  $V = R_{\lambda, i+1}^{(n)}$ . Then  $\theta^V(\mathbf{d}) = p_{\lambda, i+n}^{\mathbf{d}} - p_{\lambda, i}^{\mathbf{d}}$  by (2.5), thus the condition  $\theta^V(\mathbf{d}) = 0$  means that  $p_{\lambda, i}^{\mathbf{d}} = p_{\lambda, i+n}^{\mathbf{d}}$ . Finally,

the condition  $c_{\mathbf{d}}^V \neq 0$  and Lemma 3.3 imply that  $\theta^{V'}(\mathbf{d}) \leq 0$  for each factor representation  $V'$  of  $V$ . The sequence (2.2) implies that  $R_{\lambda,j+1}^{(n+i-j)}$  is a factor representation of  $V$  for each  $j \in [i+1, i+n-1]$ , hence the claim follows.  $\square$

Now we show that the representations described in the above lemma give rise to non-zero semi-invariants.

**Lemma 4.4.** *Let  $\lambda \in \mathbb{P}_{\mathbb{k}}^1$ ,  $i \in \mathbb{Z}$  and  $n \in \mathbb{N}$  be such that  $p_{\lambda,i}^{\mathbf{d}} = p_{\lambda,i+n}^{\mathbf{d}}$  and  $p_{\lambda,j}^{\mathbf{d}} \geq p_{\lambda,i+n}^{\mathbf{d}}$  for each  $j \in [i+1, i+n-1]$ . If  $V := R_{\lambda,i+1}^{(n)}$ , then  $\theta^V(\mathbf{d}) = 0$  and there exists  $R \in \mathcal{R}(\mathbf{d})$  such that  $c_{\mathbf{d}}^V(R) \neq 0$ .*

*Proof.* We only need to show that there exists  $R \in \mathcal{R}(\mathbf{d})$  such that  $c_{\mathbf{d}}^V(R) \neq 0$ . Let  $W \in \text{add } \mathcal{R}$  be an ext-minimal representation for  $\mathbf{d} - p \cdot \mathbf{h}$  and fix  $\mu \in \mathbb{P}_{\mathbb{k}}^1$  different from  $\lambda$  such that  $r_{\mu} = 1$ . If  $R := W \oplus R_{\mu,0}^{(p)}$ , then  $R \in \text{rep}_{\Delta}(\mathbf{d})$  and  $\text{Hom}_{\Delta}(V, R) = \text{Hom}_{\Delta}(V, W) = 0$  by Lemma 2.1, hence the claim follows.  $\square$

As a consequence we present a smaller generating set of  $\text{SI}[\Delta, \mathbf{d}]$ . First we introduce some notation. For  $\lambda \in \mathbb{P}_{\mathbb{k}}^1$  we denote by  $\mathcal{I}_{\lambda}$  the set of  $i \in [0, r_{\lambda} - 1]$  such that there exists  $n \in \mathbb{N}_+$  with  $p_{\lambda,i}^{\mathbf{d}} = p_{\lambda,i+n}^{\mathbf{d}}$  and  $p_{\lambda,j}^{\mathbf{d}} > p_{\lambda,i}^{\mathbf{d}}$  for each  $j \in [i+1, i+n-1]$  (such  $n$ , if exists, is uniquely determined by  $\lambda$  and  $i$ , and we denote it by  $n_{\lambda,i}$ ). Observe that  $\mathcal{I}_{\lambda} = \{0\}$  and  $n_{\lambda,0} = 1$  if  $r_{\lambda} = 1$ .

**Corollary 4.5.** *The algebra  $\text{SI}[\Delta, \mathbf{d}]$  is generated by the semi-invariants  $c_{\mathbf{d}}^{R_{\lambda,i+1}^{(n_{\lambda,i})}}$  for  $\lambda \in \mathbb{P}_{\mathbb{k}}^1$  and  $i \in \mathcal{I}_{\lambda}$ .*

*Proof.* For  $\lambda \in \mathbb{P}_{\mathbb{k}}^1$  we denote by  $\mathcal{I}'_{\lambda}$  the set of all pairs  $(i, n) \in [0, r_{\lambda} - 1] \times \mathbb{N}_+$  such that  $p_{\lambda,i}^{\mathbf{d}} = p_{\lambda,i+n}^{\mathbf{d}}$  and  $p_{\lambda,j}^{\mathbf{d}} \geq p_{\lambda,i}^{\mathbf{d}}$  for each  $j \in [i+1, i+n-1]$ . Observe that if  $\lambda \in \mathbb{P}_{\mathbb{k}}^1$  and  $(i, n) \in \mathcal{I}'_{\lambda}$ , then  $i \in \mathcal{I}_{\lambda}$ . Corollary 3.8 and Lemma 4.3 imply that the algebra  $\text{SI}[\Delta, \mathbf{d}]$  is generated by the semi-invariants  $c_{\mathbf{d}}^{R_{\lambda,i+1}^{(n)}}$  for  $\lambda \in \mathbb{P}_{\mathbb{k}}^1$  and  $(i, n) \in \mathcal{I}'_{\lambda}$ . Now, let  $\lambda \in \mathbb{P}_{\mathbb{k}}^1$  and  $(i, n) \in \mathcal{I}'_{\lambda}$ . Obviously,  $n \geq n_{\lambda,i}$ . If  $n > n_{\lambda,i}$ , then (up to a non-zero scalar)  $c_{\mathbf{d}}^{R_{\lambda,i+1}^{(n)}} = c_{\mathbf{d}}^{R_{\lambda,i+1}^{(n_{\lambda,i})}} \cdot c_{\mathbf{d}}^{R_{\lambda,i+n_{\lambda,i}+1}^{(n-n_{\lambda,i})}}$  by Corollary 3.2(1), as according to (2.2) we have an exact sequence

$$0 \rightarrow R_{\lambda,i+1}^{(n_{\lambda,i})} \rightarrow R_{\lambda,i+1}^{(n)} \rightarrow R_{\lambda,i+n_{\lambda,i}+1}^{(n-n_{\lambda,i})} \rightarrow 0.$$

Since  $R_{\lambda,i+n_{\lambda,i}+1}^{(n-n_{\lambda,i})} = R_{\lambda,(i+n_{\lambda,i}+1) \bmod r_{\lambda}}^{(n-n_{\lambda,i})}$  and  $((i+n_{\lambda,i}) \bmod r_{\lambda}, n-n_{\lambda,i}) \in \mathcal{I}'_{\lambda}$ , the claim follows by induction.  $\square$

At the later stage we will prove that for each non-zero semi-invariant  $f$  there exists  $R \in \mathcal{R}(\mathbf{d})$  such that  $f(R) \neq 0$ . At the moment we formulate the following versions of this fact.

**Lemma 4.6.** *Let  $V$  be a representation of  $\Delta$  such that  $\theta^V(\mathbf{d}) = 0$  and  $c_{\mathbf{d}}^V \neq 0$ . Then there exists  $R \in \mathcal{R}(\mathbf{d})$  such that  $c_{\mathbf{d}}^V(R) \neq 0$ .*

*Proof.* Let  $X$  be an indecomposable direct summand of  $V$ . Lemma 3.4 implies that  $c_{\mathbf{d}}^X \neq 0$ . Consequently, Lemmas 4.3 and 4.4 imply that there exists  $R_X \in \mathcal{R}(\mathbf{d})$  such that  $c_{\mathbf{d}}^X(R_X) \neq 0$ . Since  $\mathcal{R}(\mathbf{d})$  is an irreducible and open subset of  $\text{rep}_{\Delta}(\mathbf{d})$  [15, Section 4], there exists  $R \in \mathcal{R}(\mathbf{d})$  such that  $c_{\mathbf{d}}^X(R) \neq 0$  for each indecomposable direct summand  $X$  of  $V$ . Using once more Lemma 3.4 we obtain that  $c_{\mathbf{d}}^V(R) \neq 0$ .  $\square$

**Lemma 4.7.** *If  $q \in \mathbb{N}$  and  $f \in \text{SI}[\Delta, \mathbf{d}]_{\langle q \cdot \mathbf{h}, - \rangle_{\Delta}}$  is non-zero, then there exists  $R \in \mathcal{R}(\mathbf{d})$  such that  $f(R) \neq 0$ .*

*Proof.* If  $q = 0$ , then the claim is obvious, since  $\text{SI}[\Delta, \mathbf{d}]_0 = \mathbb{k}$ . Thus assume  $q > 0$ . We know that  $\text{SI}[\Delta, \mathbf{d}]_{\langle q \cdot \mathbf{h}, - \rangle_{\Delta}}$  is spanned by the functions  $c_{\mathbf{d}}^V$  for  $V \in \text{add } \mathcal{R}$  with dimension vector  $q \cdot \mathbf{h}$ . It is enough to prove that  $c_{\mathbf{d}}^V(M) = 0$  for all  $V \in \text{add } \mathcal{R}$  and  $M \in \text{rep}_{\Delta}(\mathbf{d})$  such that  $\mathbf{dim } V = q \cdot \mathbf{h}$  and  $M \notin \mathcal{R}(\mathbf{d})$ . Every such  $M$  has an indecomposable direct summand  $Q$  from  $\mathcal{Q}$ . Indeed, since  $M \notin \mathcal{R}(\mathbf{d})$ , it has an indecomposable direct summand  $X$  which belongs to  $\mathcal{P} \cup \mathcal{Q}$ . If  $X \in \mathcal{Q}$ , then we take  $Q := X$ . If  $X \in \mathcal{P}$ , then  $\langle \mathbf{dim } M - \mathbf{dim } X, \mathbf{h} \rangle_{\Delta} < 0$  by (2.3) and (2.6). Consequently,  $M$  has an indecomposable direct summand  $Q$  with  $\langle \mathbf{dim } Q, \mathbf{h} \rangle_{\Delta} < 0$ . Using again (2.3) and (2.6) we get  $Q \in \mathcal{Q}$ . Then

$$\dim_{\mathbb{k}} \text{Hom}_{\Delta}(V, M) \geq \dim_{\mathbb{k}} \text{Hom}_{\Delta}(V, Q) = \langle q \cdot \mathbf{h}, \mathbf{dim } Q \rangle_{\Delta} > 0$$

by (2.4) and the claim follows.  $\square$

Recall from Corollary 4.2 that the possible weights are of the form  $\langle \mathbf{r}, - \rangle_{\Delta}$  for  $\mathbf{r} \in \mathbf{R}$  such that  $\langle \mathbf{r}, \mathbf{d} \rangle_{\Delta} = 0$ . Our next aim is to show that it is enough to understand those which are for the form  $\langle q \cdot \mathbf{h}, - \rangle_{\Delta}$  for  $q \in \mathbb{N}$ .

We start with the following easy lemma.

**Lemma 4.8.** *Let  $W \in \text{add } \mathcal{R}$  be such that  $\theta^W(\mathbf{d}) = 0$  and  $c_{\mathbf{d}}^W \neq 0$ . If  $q \in \mathbb{N}$  and  $f \in \text{SI}[\Delta, \mathbf{d}]_{\langle q \cdot \mathbf{h}, - \rangle_{\Delta}}$  is non-zero, then there exists  $R \in \mathcal{R}(\mathbf{d})$  such that  $c_{\mathbf{d}}^W(R) \cdot f(R) \neq 0$ .*

*Proof.* Since  $\mathcal{R}(\mathbf{d})$  is an open irreducible subset of  $\text{rep}_{\Delta}(\mathbf{d})$ , the claim follows from Lemmas 4.6 and 4.7.  $\square$

**Proposition 4.9.** *Let  $\mathbf{r} \in \mathbf{R}$ ,  $\langle \mathbf{r}, \mathbf{d} \rangle_{\Delta} = 0$  and  $W \in \text{add } \mathcal{R}$  be an ext-minimal representation for  $\mathbf{r} - p^{\mathbf{r}} \cdot \mathbf{h}$ .*

- (1) *If  $c_{\mathbf{d}}^W = 0$ , then  $\text{SI}[\Delta, \mathbf{d}]_{\langle \mathbf{r}, - \rangle_{\Delta}} = 0$ .*
- (2) *If  $c_{\mathbf{d}}^W \neq 0$ , then the map*

$$\text{SI}[\Delta, \mathbf{d}]_{\langle p^{\mathbf{r}} \cdot \mathbf{h}, - \rangle_{\Delta}} \rightarrow \text{SI}[\Delta, \mathbf{d}]_{\langle \mathbf{r}, - \rangle_{\Delta}}, f \mapsto c_{\mathbf{d}}^W \cdot f,$$

*is an isomorphism of vector spaces.*

*Proof.* Let  $\Phi : \text{SI}[\Delta, \mathbf{d}]_{\langle p^{\mathbf{r}} \cdot \mathbf{h}, - \rangle_{\Delta}} \rightarrow \text{SI}[\Delta, \mathbf{d}]_{\langle \mathbf{r}, - \rangle_{\Delta}}$  be the map given by  $\Phi(f) := c_{\mathbf{d}}^W \cdot f$ , for  $f \in \text{SI}[\Delta, \mathbf{d}]_{\langle p^{\mathbf{r}} \cdot \mathbf{h}, - \rangle_{\Delta}}$ .

It follows from Corollary 3.7 and Lemma 4.1 that  $\text{SI}[\Delta, \mathbf{d}]_{\langle \mathbf{r}, - \rangle_{\Delta}}$  is spanned by the functions  $c_{\mathbf{d}}^V$  for ext-minimal  $V \in \text{add } \mathcal{R}$  such that  $\mathbf{dim} V = \mathbf{r}$ . If  $V \in \text{add } \mathcal{R}$  is ext-minimal and  $\mathbf{dim} V = \mathbf{r}$ , then there exists an exact sequence  $0 \rightarrow R \rightarrow V \rightarrow W \rightarrow 0$ , where  $R \in \text{add } \mathcal{R}$  and  $\mathbf{dim} R = p^{\mathbf{r}} \cdot \mathbf{h}$ . Thus Corollary 3.2(1) implies that (up to a non-zero scalar)  $c_{\mathbf{d}}^V = c_{\mathbf{d}}^W \cdot c_{\mathbf{d}}^R = \Phi(c_{\mathbf{d}}^R)$ . This shows that  $\Phi$  is an epimorphism. In particular,  $\text{SI}[\Delta, \mathbf{d}]_{\langle \mathbf{r}, - \rangle_{\Delta}} = 0$  if  $c_{\mathbf{d}}^W = 0$ . On the other hand, if  $c_{\mathbf{d}}^W \neq 0$ , then  $\Phi$  is a monomorphism (hence an isomorphism) by Lemma 4.8.  $\square$

In the previous papers on the subject the authors have studied either the semi-invariants on the whole variety  $\text{rep}_{\Delta}(\mathbf{d})$  [14, 15] or on the closure of  $\mathcal{R}(\mathbf{d})$  only [29]. However, the answers they have obtained did not differ. We have the following explanation of this phenomena.

**Proposition 4.10.** *If  $f \in \mathbb{k}[\text{rep}_{\Delta}(\mathbf{d})]$  is a non-zero semi-invariant, then there exists  $R \in \mathcal{R}(\mathbf{d})$  such that  $f(R) \neq 0$ .*

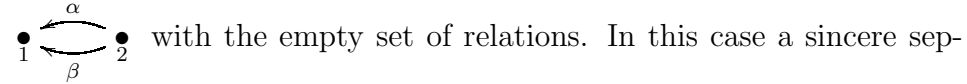
*Proof.* Fix  $\mathbf{r} \in \mathbf{R}$  such that  $f \in \text{SI}[\Delta, \mathbf{d}]_{\langle \mathbf{r}, - \rangle_{\Delta}}$ . The previous lemma implies that  $f = c_{\mathbf{d}}^W \cdot f'$ , where  $W \in \text{add } \mathcal{R}$  is an ext-minimal representation with dimension vector  $\mathbf{r} - p^{\mathbf{r}} \cdot \mathbf{h}$  and  $f' \in \text{SI}[\Delta, \mathbf{d}]_{\langle p^{\mathbf{r}} \cdot \mathbf{h}, - \rangle_{\Delta}}$ . Consequently, the claim follows from Lemma 4.8.  $\square$

Observe that this proposition means in particular, that  $\text{SI}[\Delta, \mathbf{d}]$  is a domain, hence the product of two non-zero semi-invariants is non-zero again.

Proposition 4.9 implies that the subalgebra  $\bigoplus_{q \in \mathbb{N}} \text{SI}[\Delta, \mathbf{d}]_{\langle q \cdot \mathbf{h}, - \rangle_{\Delta}}$  of  $\text{SI}[\Delta, \mathbf{d}]$  plays a crucial role. In Section 6 we show that the study of this subalgebra can be reduced to the case of the Kronecker quiver. Thus in the next section we recall facts about semi-invariants for the Kronecker quiver.

## 5. THE KRONECKER QUIVER

Our aim in this section is to collect necessary facts about representations and semi-invariants for the Kronecker quiver  $K_2$ , i.e. the quiver



with the empty set of relations. In this case a sincere separating exact subcategory is uniquely determined. Let  $\mathcal{T} = \coprod_{\lambda \in \mathbb{P}_{\mathbb{k}}^1} \mathcal{T}_{\lambda}$  by the sincere separating exact subcategory of  $\text{ind } K_2$ .

For  $\zeta, \xi \in \mathbb{k}$  let  $N_{\zeta, \xi}$  be the representation  $\mathbb{k} \begin{matrix} \xleftarrow{\zeta} \\ \xrightarrow{\xi} \end{matrix} \mathbb{k}$ . Then the sim-

ple objects in  $\text{add } \mathcal{T}$  are precisely the representations  $N_{\zeta, \xi}$  for  $(\zeta : \xi) \in \mathbb{P}_{\mathbb{k}}^1$ . Moreover, if  $(\zeta : \xi), (\zeta' : \xi') \in \mathbb{P}_{\mathbb{k}}^1$ , then  $N_{\zeta, \xi} \simeq N_{\zeta', \xi'}$  if and only if  $(\zeta : \xi) = (\zeta' : \xi')$ . Consequently, by abuse of notation, we will denote

$N_{\zeta, \xi}$  by  $N_{(\zeta : \xi)}$  for  $(\zeta : \xi) \in \mathbb{P}_k^1$ . By choosing our parameterization appropriately we may assume that  $N_\lambda \in \mathcal{T}_\lambda$  for each  $\lambda \in \mathbb{P}_k^1$ . In particular,  $\tau N_\lambda = N_\lambda$  for each  $\lambda \in \mathbb{P}_k^1$ .

The Kronecker quiver can be viewed as the minimal concealed-canonical bound quiver. Namely, we can embed the category  $\text{rep } K_2$  into the category of representations of an arbitrary concealed-canonical quiver. We describe a construction of such an embedding more precisely.

Let  $\Delta$  be a concealed-canonical bound quiver with a sincere separating exact subcategory  $\mathcal{R}$  of  $\text{ind } \Delta$ . Let  $R := \bigoplus_{\lambda \in \mathbb{P}_k^1} \bigoplus_{i \in I_\lambda} R_{\lambda, i}$  for subsets  $I_\lambda \subseteq [0, r_\lambda - 1]$  such that  $|I_\lambda| = r_\lambda - 1$  (in particular,  $I_\lambda = \emptyset$  if  $r_\lambda = 1$ ), where we use notation introduced in Section 2. Let  $R^\perp$  denote the full subcategory of  $\text{rep } \Delta$ , whose objects are  $M \in \text{rep } \Delta$  such that  $\text{Hom}_\Delta(R, M) = 0 = \text{Ext}_\Delta^1(R, M)$ . Lenzing and de la Peña [23, Proposition 4.2] have proved that there exists a fully faithful exact functor  $F : \text{rep } K_2 \rightarrow \text{rep } \Delta$  which induces an equivalence between  $\text{rep } K_2$  and  $R^\perp$ . Moreover,  $F$  induces an equivalence between  $\mathcal{T}$  and  $R^\perp \cap \mathcal{R}$ . The simple objects in  $R^\perp \cap (\text{add } \mathcal{R})$ , which are the images of the simple objects in  $\text{add } \mathcal{T}$ , are of the form  $R_{\lambda, i_\lambda}^{(r_\lambda)}$  for  $\lambda \in \mathbb{P}_k^1$ , where for  $\lambda \in \mathbb{P}_k^1$  we denote by  $i_\lambda$  the unique element of  $[0, r_\lambda - 1] \setminus I_\lambda$ . Consequently, (if we choose appropriate parameterization)  $F(N_\lambda) \simeq R_{\lambda, i_\lambda}^{(r_\lambda)}$  for each  $\lambda \in \mathbb{P}_k^1$ .

Let  $p \in \mathbb{N}$ . We define the functions  $f_{(p,p)}^{(0)}, \dots, f_{(p,p)}^{(p)} \in \mathbb{k}[\text{rep}_{K_2}(p, p)]$  by the condition: if  $V \in \text{rep}_{K_2}(p, p)$ , then

$$\det(S \cdot V_\alpha - T \cdot V_\beta) = \sum_{i \in [0, p]} S^i \cdot T^{p-i} \cdot f_{(p,p)}^{(i)}(V).$$

Note that  $f_{(p,p)}^{(0)}, \dots, f_{(p,p)}^{(p)}$  are semi-invariants of weight  $(-1, 1)$ . If  $(\zeta : \xi) \in \mathbb{P}_k^1$ , then (by choosing a projective presentation of  $N_{\zeta, \xi}$  in an appropriate way) we get

$$(5.1) \quad c_{(p,p)}^{N_{\zeta, \xi}}(V) = \det(\xi \cdot V_\alpha - \zeta \cdot V_\beta) = \sum_{i \in [0, p]} \xi^i \cdot \zeta^{p-i} \cdot f_{(p,p)}^{(i)}(V).$$

It is well known (see for example [30]) that  $\text{SI}[K_2, (p, p)]$  is the polynomial algebra in  $f_{(p,p)}^{(0)}, \dots, f_{(p,p)}^{(p)}$ . In particular,

$$(5.2) \quad \dim_{\mathbb{k}} \text{SI}[K_2, (p, p)]_{(-q, q)} = \binom{q+p}{q}$$

for each  $q \in \mathbb{N}$ .

We will need the following lemma.

**Lemma 5.1.** *If  $f_1, f_2 \in \text{SI}[K_2, (p, p)]_{(-1, 1)}$  and*

$$\{V \in \text{rep}_{K_2}(p, p) : f_1(V) = 0\} = \{V \in \text{rep}_{K_2}(p, p) : f_2(V) = 0\},$$

*then (up to a non-zero scalar)  $f_1 = f_2$ .*

*Proof.* From the description of  $\text{SI}[K_2, (p, p)]$  it follows that  $f_1$  and  $f_2$  are irreducible, hence the claim follows.  $\square$

## 6. THE MAIN RESULT

Throughout this section we fix a concealed-canonical bound quiver  $\Delta$  and a sincere separating exact subcategory  $\mathcal{R}$  of  $\text{ind } \Delta$ . We use freely notation introduced in Section 2. We also fix  $\mathbf{d} \in \mathbf{R}$  such that  $p := p^{\mathbf{d}} > 0$ .

First we investigate the algebra  $\bigoplus_{q \in \mathbb{N}} \text{SI}[\Delta, \mathbf{d}]_{\langle q \cdot \mathbf{h}, - \rangle_{\Delta}}$ . We introduce some notation. For  $\lambda \in \mathbb{P}_{\mathbb{k}}^1$  we denote by  $\mathcal{I}_{\lambda}^0$  the set of  $i \in [0, r_{\lambda} - 1]$  such that  $p_{\lambda, i}^{\mathbf{d}} = 0$ . Observe that  $\mathcal{I}_{\lambda}^0 \subseteq \mathcal{I}_{\lambda}$  for each  $\lambda \in \mathbb{P}_{\mathbb{k}}^1$  (the sets  $\mathcal{I}_{\lambda}$  for  $\lambda \in \mathbb{P}_{\mathbb{k}}^1$  were introduced before Corollary 4.5). Recall that, for  $\lambda \in \mathbb{P}_{\mathbb{k}}^1$  and  $i \in \mathcal{I}_{\lambda}$ ,  $n_{\lambda, i}$  denotes the minimal  $n \in \mathbb{N}_+$  such that  $p_{\lambda, i+n}^{\mathbf{d}} = 0$ .

We put  $c_{\mathbf{d}}^{\lambda} := \prod_{i \in \mathcal{I}_{\lambda}^0} c_{\mathbf{d}}^{R_{\lambda, i}^{(n_{\lambda, i})}}$ . An iterated application of Corollary 3.2(1) to exact sequences of the form (2.2) implies that  $c_{\mathbf{d}}^{\lambda} = c_{\mathbf{d}}^{R_{\lambda, i}^{(r_{\lambda})}}$  for each  $i \in \mathcal{I}_{\lambda}^0$ .

We have the following fact.

**Lemma 6.1.** *The algebra  $\bigoplus_{q \in \mathbb{N}} \text{SI}[\Delta, \mathbf{d}]_{\langle q \cdot \mathbf{h}, - \rangle_{\Delta}}$  is generated by the semi-invariants  $c_{\mathbf{d}}^{\lambda}$  for  $\lambda \in \mathbb{X}$ .*

*Proof.* This fact has been proved in [4], but for completeness we include its (shorter) proof here.

Fix  $q \in \mathbb{N}$ . Proposition 3.7 and Lemma 4.1 imply that  $\text{SI}[\Delta, \mathbf{d}]_{\langle q \cdot \mathbf{h}, - \rangle_{\Delta}}$  is spanned by the semi-invariants  $c_{\mathbf{d}}^V$  for ext-minimal  $V \in \text{add } \mathcal{R}$  with dimension vector  $q \cdot \mathbf{h}$ . Fix such  $V$ . Since  $V$  is ext-minimal with dimension vector  $q \cdot \mathbf{h}$ ,  $V = \bigoplus_{\lambda \in \mathbb{X}} R_{\lambda, i_{\lambda}}^{(k_{\lambda} \cdot r_{\lambda})}$ , where  $\mathbb{X} \subseteq \mathbb{P}_{\mathbb{k}}^1$  and  $i_{\lambda} \in [0, r_{\lambda} - 1]$  and  $k_{\lambda} \in \mathbb{N}_+$  for each  $\lambda \in \mathbb{X}$ . Moreover, Lemma 4.3 implies that  $i_{\lambda} \in \mathcal{I}_{\lambda}^0$  for each  $\lambda \in \mathbb{X}$ . An iterated application of Corollary 3.2(1) to exact sequences of the form (2.2) implies that  $c_{\mathbf{d}}^{R_{\lambda, i_{\lambda}}^{(k_{\lambda} \cdot r_{\lambda})}} = (c_{\mathbf{d}}^{\lambda})^{k_{\lambda}}$  for each  $\lambda \in \mathbb{X}$ . Consequently,  $c_{\mathbf{d}}^V = \prod_{\lambda \in \mathbb{X}} (c_{\mathbf{d}}^{\lambda})^{k_{\lambda}}$  by Lemma 3.4, hence the claim follows.  $\square$

The following fact is crucial.

**Proposition 6.2.** *There exists a regular map*

$$\Phi : \text{rep}_{K_2}(p, p) \rightarrow \text{rep}_{\Delta}(\mathbf{d})$$

*such that  $\Phi^*$  induces an isomorphism*

$$\bigoplus_{q \in \mathbb{N}} \text{SI}[\Delta, \mathbf{d}]_{\langle q \cdot \mathbf{h}, - \rangle_{\Delta}} \rightarrow \bigoplus_{q \in \mathbb{N}} \text{SI}[K_2, (p, p)]_{(-q, q)}$$

*of  $\mathbb{N}$ -graded rings and (up to a non-zero scalar)  $\Phi^*(c_{\mathbf{d}}^{\lambda}) = c_{(p, p)}^{N_{\lambda}}$  for each  $\lambda \in \mathbb{P}_{\mathbb{k}}^1$ .*



*Proof.* For each  $\lambda \in \mathbb{P}_{\mathbb{k}}^1$  we fix  $i_\lambda \in \mathcal{I}_\lambda^0$ . From Section 5 we know that there exists a fully faithful exact functor  $F : \text{rep } K_2 \rightarrow \text{rep } \Delta$  such that  $F(N_\lambda) \simeq R_{\lambda, i_\lambda}^{(r_\lambda)}$  for each  $\lambda \in \mathbb{P}_{\mathbb{k}}^1$ . Observe that for each  $R \in \text{add}(\coprod_{\lambda \in \mathbb{P}_{\mathbb{k}}^1 \setminus \mathbb{X}_0} \mathcal{R}_\lambda)$  (recall that  $\mathbb{X}_0$  is the set of all  $\lambda \in \mathbb{P}_{\mathbb{k}}^1$  such that  $r_\lambda > 1$ ) there exists  $N \in \mathcal{T}$  with  $F(N) \simeq R$ .

Put  $E_1 := F(S_1)$  and  $E_2 := F(S_2)$ , where  $S_i$  is the simple representation of  $K_2$  at  $i$ , for  $i \in \{1, 2\}$ , i.e.

$$S_1 := \mathbb{k} \begin{array}{c} \longleftarrow \\ \longrightarrow \end{array} 0 \quad \text{and} \quad S_2 := 0 \begin{array}{c} \longleftarrow \\ \longrightarrow \end{array} \mathbb{k}.$$

Then [24, Proposition 2.3] (see also [9, Proposition 5.2]) implies that there exists a regular map  $\Phi' : \text{rep}_{K_2}(p, p) \rightarrow \text{rep}_\Delta(p \cdot \mathbf{h})$  such that  $\Phi'(N) \simeq F(N)$  for each  $N \in \text{rep}_{K_2}(p, p)$ . Moreover, there exists a morphism  $\varphi : \text{GL}(p, p) \rightarrow \text{GL}(p \cdot \mathbf{h})$  of algebraic groups such that  $\Phi'(g * N) = \varphi(g) * \Phi'(N)$ , for all  $g \in \text{GL}(p, p)$  and  $N \in \text{rep}_\Delta(p \cdot \mathbf{h})$ , and

$$(6.1) \quad \chi^\theta(\varphi(g)) = (\det(g(1))^{\theta(\dim E_1)} \cdot (\det(g(2)))^{\theta(\dim E_2)}),$$

for all  $g \in \text{GL}(p, p)$  and  $\theta \in \mathbb{Z}^{\Delta_0}$ .

Let  $W \in \text{add } \mathcal{R}$  be an ext-minimal representation for  $\mathbf{d}' := \mathbf{d} - p \cdot \mathbf{h}$ . We define  $\Phi : \text{rep}_{K_2}(p, p) \rightarrow \text{rep}_\Delta(\mathbf{d})$  by  $\Phi(N) := \Phi'(N) \oplus W$  for  $N \in \text{rep}_{K_2}(p, p)$ .

Let  $q \in \mathbb{N}$ . We show that  $\Phi^*(f)$  is a semi-invariant of weight  $(-q, q)$  for each  $f \in \text{SI}[\Delta, \mathbf{d}]_{\langle q \cdot \mathbf{h}, - \rangle_\Delta}$ . Using Proposition 3.6 and Lemma 4.1 it suffices to show that  $\Phi^*(c^V)$  is a semi-invariant of weight  $(-q, q)$  for each representation  $V$  of  $\Delta$  with dimension vector  $q \cdot \mathbf{h}$ . Now, if  $g \in \text{GL}(p, p)$  and  $N \in \text{rep}_{K_2}(p, p)$ , then

$$\begin{aligned} (\Phi^*(c_{\mathbf{d}}^V))(g^{-1} * N) &= c_{\mathbf{d}}^V(W \oplus \Phi'(g^{-1} * N)) \\ &= c_{\mathbf{d}'}^V(W) \cdot c_{p \cdot \mathbf{h}}^V(\varphi(g^{-1}) * \Phi'(N)) \\ &= c_{\mathbf{d}'}^V(W) \cdot \chi^{\langle q \cdot \mathbf{h}, - \rangle_\Delta}(\varphi(g)) \cdot c_{p \cdot \mathbf{h}}^V(\Phi(N)) \\ &= \chi^{\langle q \cdot \mathbf{h}, - \rangle_\Delta}(\varphi(g)) \cdot (\Phi^*(c_{\mathbf{d}}^V))(N), \end{aligned}$$

where the second and the last equalities follow from Lemma 3.5. Using (6.1) we get

$$\chi^{\langle q \cdot \mathbf{h}, - \rangle_\Delta}(\varphi(g)) = (\det(g(1))^{-q} \cdot (\det(g(2)))^q,$$

since

$$\langle \mathbf{h}, \dim E_i \rangle_\Delta = \langle (1, 1), \dim S_i \rangle_{K_2} = (-1)^i$$

for each  $i \in \{1, 2\}$  (we use here that  $F$  is exact).

The above implies that  $\Phi^*$  induces a homomorphism

$$(6.2) \quad \bigoplus_{q \in \mathbb{N}} \text{SI}[\Delta, \mathbf{d}]_{\langle q \cdot \mathbf{h}, - \rangle_\Delta} \rightarrow \bigoplus_{q \in \mathbb{N}} \text{SI}[K_2, (p, p)]_{(-q, q)}$$

of  $\mathbb{N}$ -graded rings. We need to show that this is an isomorphism.

First we show  $\Phi^*(f) \neq 0$  for each non-zero semi-invariant  $f$  (in particular, this will imply that (6.2) is a monomorphism). Let

$$\mathcal{Z} := \{M \in \text{rep}_{\Delta}(\mathbf{d}) : \text{there exists } N \in \text{rep}_{K_2}(p, p) \text{ such that } M \simeq W \oplus \Phi(N)\}.$$

In other words,  $\mathcal{Z}$  is in the closure of the image of  $\Phi$  under the action of  $\text{GL}(\mathbf{d})$ . Using Proposition 4.10 it suffices to show that  $\mathcal{Z}$  contains a non-empty open subset of  $\mathcal{R}(\mathbf{d})$ . Let

$$\mathcal{U} := \{M \in \mathcal{R}(\mathbf{d}) : c_{\mathbf{d}}^{\lambda}(M) \neq 0 \text{ for each } \lambda \in \mathbb{X}_0 \text{ and } \dim_{\mathbb{k}} \text{End}_{\Delta}(M) = p + \langle \mathbf{d}, \mathbf{d} \rangle_{\Delta}\}.$$

Since the function

$$\text{rep}_{\Delta}(\mathbf{d}) \ni M \mapsto \dim_{\mathbb{k}} \text{End}_{\Delta}(M) \in \mathbb{Z}$$

is upper semi-continuous, (2.7) implies that  $\mathcal{U}$  is a non-empty open subset of  $\mathcal{R}(\mathbf{d})$ , which consists of ext-minimal representations. In particular, if  $M \in \mathcal{U}$ , then there exists an exact sequence of the form  $0 \rightarrow R \rightarrow M \rightarrow W \rightarrow 0$  with  $R \in \text{add } \mathcal{R}$  such that  $\mathbf{dim} R = p \cdot \mathbf{h}$ . If  $p_{\lambda}^R \neq 0$  for some  $\lambda \in \mathbb{X}_0$ , then  $\text{Hom}_{\Delta}(R_{\lambda, i_{\lambda}}^{(r_{\lambda})}, R) \neq 0$ . Consequently,  $\text{Hom}_{\Delta}(R_{\lambda, i_{\lambda}}^{(r_{\lambda})}, M) \neq 0$ , hence  $c_{\mathbf{d}}^{\lambda}(M) = 0$ , contradiction. Thus  $p_{\lambda}^R = 0$  for each  $\lambda \in \mathbb{X}_0$ , hence  $M \simeq W \oplus R$  and  $R \in \text{add}(\coprod_{\lambda \in \mathbb{P}_{\mathbb{k}}^1 \setminus \mathbb{X}_0} \mathcal{R}_{\lambda})$ . In particular, there exists  $N \in \text{rep } \mathcal{T}$  such that  $F(N) \simeq R$ , hence  $M \in \mathcal{Z}$ .

Now we fix  $\lambda \in \mathbb{P}_{\mathbb{k}}^1$ . We show that (up to a non-zero scalar)  $\Phi^*(c_{\mathbf{d}}^{\lambda}) = c_{(p,p)}^{N_{\lambda}}$  for each  $\lambda \in \mathbb{P}_{\mathbb{k}}^1$ . According to Lemma 6.1 this will imply that (6.2) is an epimorphism, hence finish the proof. Fix  $N \in \text{rep}_{K_2}(p, p)$ . Then

$$(\Phi^*(c_{\mathbf{d}}^{\lambda}))(N) = 0 \text{ if and only if } \text{Hom}_{\Delta}(R_{\lambda, i_{\lambda}}^{(r_{\lambda})}, F(N)) \neq 0.$$

Since  $R_{\lambda, i_{\lambda}}^{(r_{\lambda})} \simeq F(N_{\lambda})$  and  $F$  is fully faithful,

$$(\Phi^*(c_{\mathbf{d}}^{\lambda}))(N) = 0 \text{ if and only if } \text{Hom}_{K_2}(N_{\lambda}, N) \neq 0.$$

Similarly, if  $N \in \text{rep}_{K_2}(p, p)$ , then

$$c_{(p,p)}^{N_{\lambda}}(N) = 0 \text{ if and only if } \text{Hom}_{K_2}(N_{\lambda}, N) \neq 0.$$

Consequently, the claim follows from Lemma 5.1.  $\square$

**Corollary 6.3.** *If  $\mathbf{r} \in \mathbf{R}$  and  $\text{SI}[\Delta, \mathbf{d}]_{\langle \mathbf{r}, - \rangle_{\Delta}} \neq 0$ , then*

$$\dim_{\mathbb{k}} \text{SI}[\Delta, \mathbf{d}]_{\langle \mathbf{r}, - \rangle_{\Delta}} = \binom{p^{\mathbf{r}} + p}{p^{\mathbf{r}}}.$$

*Proof.* Proposition 4.9(2) implies that

$$\dim_{\mathbb{k}} \text{SI}[\Delta, \mathbf{d}]_{\langle \mathbf{r}, - \rangle_{\Delta}} = \dim_{\mathbb{k}} \text{SI}[\Delta, \mathbf{d}]_{\langle p^{\mathbf{r}} \cdot \mathbf{h}, - \rangle_{\Delta}}.$$

Next,

$$\dim_{\mathbb{k}} \text{SI}[\Delta, \mathbf{d}]_{\langle p^{\mathbf{r}} \cdot \mathbf{h}, - \rangle_{\Delta}} = \dim_{\mathbb{k}} \text{SI}[K_2, (p, p)]_{(-p^{\mathbf{r}}, p^{\mathbf{r}})}$$

by Proposition 6.2, hence the claim follows from (5.2).  $\square$

Let  $\Phi : \text{rep}_{K_2}(p, p) \rightarrow \text{rep}_{\Delta}(\mathbf{d})$  be a regular map constructed in Proposition 6.2. For  $j \in [0, p]$  we denote by  $f_{\mathbf{d}}^{(j)}$  the inverse image of  $f_{(p,p)}^{(j)}$  under  $\Phi^*$ . Then (5.1) implies that (up to a non-zero scalar)

$$(6.3) \quad c_{\mathbf{d}}^{(\zeta:\xi)} = \sum_{j \in [0, p]} \xi^j \cdot \zeta^{p-j} \cdot f_{\mathbf{d}}^{(j)}$$

for each  $(\zeta : \xi) \in \mathbb{P}_{\mathbb{k}}^1$ . As the first application we get the following (smaller) set of generators of  $\text{SI}[\Delta, \mathbf{d}]$ .

**Proposition 6.4.** *The algebra  $\text{SI}[\Delta, \mathbf{d}]$  is generated by the semi-invariants  $f_{\mathbf{d}}^{(0)}, \dots, f_{\mathbf{d}}^{(p)}$  and  $c_{\mathbf{d}}^{R_{\lambda, i+1}^{(n_{\lambda, i})}}$  for  $\lambda \in \mathbb{X}_0$  and  $i \in \mathcal{I}_{\lambda}$ .*

*Proof.* Recall from Corollary 4.5 that the algebra  $\text{SI}[\Delta, \mathbf{d}]$  is generated by the semi-invariants  $c_{\mathbf{d}}^{R_{\lambda, i+1}^{(n_{\lambda, i})}}$  for  $\lambda \in \mathbb{P}_{\mathbb{k}}^1$  and  $i \in \mathcal{I}_{\lambda}$ . Thus we only need to express, for each  $\lambda \in \mathbb{P}_{\mathbb{k}}^1 \setminus \mathbb{X}_0$  and  $i \in \mathcal{I}_{\lambda}$ ,  $c_{\mathbf{d}}^{R_{\lambda, i+1}^{(n_{\lambda, i})}}$  as the polynomial in the semi-invariants listed in the proposition. However, if  $\lambda \in \mathbb{P}_{\mathbb{k}}^1 \setminus \mathbb{X}_0$  and  $i \in \mathcal{I}_{\lambda}$ , then  $c_{\mathbf{d}}^{R_{\lambda, i+1}^{(n_{\lambda, i})}} = c_{\mathbf{d}}^{\lambda}$ , hence the claim follows from (6.3).  $\square$

We give another formulation of Proposition 6.4. Let  $\mathcal{A}$  be the polynomial algebra in the indeterminates  $S_0, \dots, S_p$  and  $T_{\lambda, i}$  for  $\lambda \in \mathbb{X}_0$  and  $i \in \mathcal{I}_{\lambda}$ . Proposition 6.4 says that the homomorphism  $\Psi : \mathcal{A} \rightarrow \text{SI}[\Delta, \mathbf{d}]$  given by the formulas:  $\Psi(S_j) := f_{\mathbf{d}}^{(j)}$ , for  $j \in [0, p]$ , and  $\Psi(T_{\lambda, i}) := c_{\mathbf{d}}^{R_{\lambda, i+1}^{(n_{\lambda, i})}}$ , for  $\lambda \in \mathbb{X}_0$  and  $i \in \mathcal{I}_{\lambda}$ , is an epimorphism. Our last aim is to describe its kernel.

First, we introduce an  $\mathbf{R}$ -grading in  $\mathcal{A}$  by specifying the degrees of the indeterminates as follows:  $\deg(S_j) := \mathbf{h}$  for  $j \in [0, p]$  and  $\deg(T_{\lambda, i}) := \mathbf{e}_{\lambda, i}^{n_{\lambda, i}}$ , for  $\lambda \in \mathbb{X}_0$  and  $i \in \mathcal{I}_{\lambda}$ . Note that  $\Psi$  is a homogenous map, i.e.  $\Psi(\mathcal{A}_{\mathbf{r}}) = \text{SI}[\Delta, \mathbf{d}]_{\langle \mathbf{r}, - \rangle_{\Delta}}$  for each  $\mathbf{r} \in \mathbf{R}$ .

Let  $\mathbf{R}_0$  be the submonoid of  $\mathbf{R}$  generated by the elements  $\mathbf{h}$  and  $\mathbf{e}_{\lambda, i}^{n_{\lambda, i}}$  for  $\lambda \in \mathbb{X}_0$  and  $i \in \mathcal{I}_{\lambda}$ . Obviously, if  $\mathbf{r} \in \mathbf{R}$ , then  $\mathcal{A}_{\mathbf{r}} \neq 0$  if and only if  $\mathbf{r} \in \mathbf{R}_0$ . Similarly, Corollary 4.5 implies that  $\text{SI}[\Delta, \mathbf{d}]_{\langle \mathbf{r}, - \rangle_{\Delta}} \neq 0$  if and only if  $\mathbf{r} \in \mathbf{R}_0$  (recall that  $\text{SI}[\Delta, \mathbf{d}]$  is a domain).

**Lemma 6.5.** *If  $\mathbf{r} \in \mathbf{R}_0$ , then*

$$\dim_{\mathbb{k}} \mathcal{A}_{\mathbf{r}} = \binom{p^{\mathbf{r}} + p + |\mathbb{X}_0|}{p^{\mathbf{r}}}.$$

*Proof.* One easily observes that there is an isomorphism  $\mathcal{A}_{p^{\mathbf{r}} \cdot \mathbf{h}} \rightarrow \mathcal{A}_{\mathbf{r}}$  of vector spaces (induced by multiplying by the unique monomial of degree  $\mathbf{r} - p^{\mathbf{r}} \cdot \mathbf{h}$ ). Moreover,  $\bigoplus_{q \in \mathbb{N}} \mathcal{A}_{q \cdot \mathbf{h}}$  is the polynomial algebra generated by  $S_0, \dots, S_p$  and  $\prod_{i \in \mathcal{I}_{\lambda}} T_{\lambda, i}$  for  $\lambda \in \mathbb{X}_0$ . Now the claim follows.  $\square$

The formula (6.3) implies that for each  $\lambda \in \mathbb{X}_0$  there exist  $\zeta_\lambda, \xi_\lambda \in \mathbb{k}$  such that

$$\prod_{i \in \mathcal{I}_\lambda^0} c_{\mathbf{d}}^{R_{\lambda, i+1}^{(n_{\lambda, i})}} = \sum_{j \in [0, p]} \xi_\lambda^j \cdot \zeta_\lambda^{p-j} \cdot f_{\mathbf{d}}^{(j)}.$$

Obviously,  $(\zeta_\lambda, \xi_\lambda) \neq (0, 0)$  and  $(\zeta_\lambda : \xi_\lambda) = \lambda$ .

**Proposition 6.6.** *We have*

$$\text{Ker } \Psi = \left( \sum_{j \in [0, p]} \xi_\lambda^j \cdot \zeta_\lambda^{p-j} \cdot S_j - \prod_{i \in \mathcal{I}_\lambda^0} T_{i, \lambda} : \lambda \in \mathbb{X}_0 \right).$$

*Proof.* Let

$$\mathcal{J} := \left( \sum_{j \in [0, p]} \xi_\lambda^j \cdot \zeta_\lambda^{p-j} \cdot S_j - \prod_{i \in \mathcal{I}_\lambda^0} T_{i, \lambda} : \lambda \in \mathbb{X}_0 \right).$$

Obviously,  $\text{Ker } \Psi \subseteq I$ . Observe that both  $\text{Ker } \Psi$  and  $\mathcal{J}$  are graded ideals (with respect to the grading introduced above). Consequently, in order to prove our claim it suffices to show that  $\dim_{\mathbb{k}} \mathcal{J}_{\mathbf{r}} = \dim_{\mathbb{k}} \text{Ker } \Psi_{\mathbf{r}}$  for each  $\mathbf{r} \in \mathbf{R}_0$ .

We already know from Lemma 6.5 and Corollary 6.3 that

$$\begin{aligned} \dim_{\mathbb{k}} \text{Ker } \Psi_{\mathbf{r}} &= \dim_{\mathbb{k}} A_{\mathbf{r}} - \dim_{\mathbb{k}} \text{SI}[\Delta, \mathbf{r}]_{\langle \mathbf{r}, - \rangle_{\Delta}} \\ &= \binom{p^{\mathbf{r}} + p + |\mathbb{X}_0|}{p^{\mathbf{r}}} - \binom{p^{\mathbf{r}} + p}{p^{\mathbf{r}}} \end{aligned}$$

for each  $\mathbf{r} \in \mathbf{R}_0$ . On the other hand, similarly as in the proof of Lemma 6.5, we show that  $\dim_{\mathbb{k}} \mathcal{J}_{\mathbf{r}} = \dim_{\mathbb{k}} \mathcal{J}_{p^{\mathbf{r}}, \mathbf{h}}$  for each  $\mathbf{r} \in \mathbf{R}_0$ . Moreover, the algebra  $\bigoplus_{q \in \mathbb{N}} (\mathcal{A}/\mathcal{J})_{q, \mathbf{h}}$  is obviously the polynomial algebra in  $p^{\mathbf{r}} + p$  indeterminates. This, together with Lemma 6.5, immediately implies our claim.  $\square$

We may summarize our considerations in the following theorem (compare [29, Theorem 1.1]).

**Theorem 6.7.** *We have the isomorphism*

$$\text{SI}[\Delta, \mathbf{d}] \simeq \mathcal{A} / \left( \sum_{j \in [0, p]} \xi_\lambda^j \cdot \zeta_\lambda^{p-j} \cdot S_j - \prod_{i \in \mathcal{I}_\lambda^0} T_{i, \lambda} : \lambda \in \mathbb{X}_0 \right).$$

*If*

$$i(\mathbf{d}) := \{ \lambda \in \mathbb{X}_0 : |\mathcal{I}_\lambda| > 1 \},$$

*then  $\text{SI}[\Delta, \mathbf{d}]$  is a complete intersection given by  $\max(0, i(\mathbf{d}) - p - 1)$  equations. In particular,  $\text{SI}[\Delta, \mathbf{d}]$  is polynomial algebra if and only if  $i(\mathbf{d}) \leq p + 1$ .*

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